Self-similar fragmentations derived from the stable tree II : splitting at nodes

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Résumé

We study a natural fragmentation process of the so-called stable tree introduced by Duquesne and Le Gall, which consists in removing the nodes of the tree according to a certain procedure that makes the fragmentation self-similar with positive index. Explicit formulas for the semigroup are given, and we provide asymptotic results. We also give an alternative construction of this fragmentation, using paths of Lévy processes, hence echoing the two alternative constructions of the standard additive coalescent by fragmenting the Brownian continuum random tree or using Brownian paths, respectively due to Aldous-Pitman and Bertoin.

Key Words. Self-similar fragmentation, stable tree, stable processes.

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1 Introduction

The goal of this paper is to investigate a Markovian fragmentation of the so-called *stable tree*. It is a model of *continuum random tree* (CRT) depending on a parameter $\alpha \in (1,2]$ that has been introduced recently by Duquesne and Le Gall [16], and which basically corresponds to a possible scaling limit as $n \to \infty$ of a size n Galton-Watson tree with given progeny distribution. The stable tree is denoted by $\mathcal{T}$. It is a random metric space with distance d, whose elements v are called *vertices*. One of these vertices is distinguished and called the *root*. This space is a tree in that for $v, w \in \mathcal{T}$, there is a unique non-self-crossing path $[[v,w]]$ from v to w in $\mathcal{T}$, whose length equals $d(v,w)$. For every $v \in \mathcal{T}$, call *height* of v in $\mathcal{T}$ and denote by $\mathrm{ht}(v)$ the distance of v to the root. The *leaves* $\mathcal{L}(\mathcal{T})$ of $\mathcal{T}$ are those vertices that do not belong to the interior of any path leading from one vertex to another, and the *skeleton* of the tree is the set $\mathcal{T} \setminus \mathcal{L}(\mathcal{T})$ of non-leaf vertices. The *branchpoints* are the vertices b so that there exist $v \neq b, w \neq b$ such that $[[\mathrm{root}, v]] \cap [[\mathrm{root}, w]] = [[\mathrm{root}, b]]$. With each realization of $\mathcal{T}$ is associated the uniform probability measure μ, called the *mass measure*, that is supported by $\mathcal{L}(\mathcal{T})$. Details are given in Section 3.

When $\alpha = 2$, the stable tree is, up to a scale factor, the Brownian CRT of Aldous [2]. It has been shown by Aldous and Pitman [3] that a certain device for logging this tree gives rise to a *fragmentation* process which is the time-reversed process of the so-called *standard additive coalescent*. The idea is as follows. The Brownian CRT $\mathcal{T}$ is described by a σ-finite *length measure* ℓ carried by the skeleton (non-leaf vertices), and a (uniform) probability measure μ on its leaves, called the mass measure. For $t \geq 0$, consider a Poisson random measure on $\mathcal{T}$ with intensity $t\ell$, in a consistent way as t varies. When the marked vertices of the tree are removed, the tree is decomposed into a random forest, whose ranked μ-masses form an element $F_{\mathrm{AP}}(t)$ of the space

$$S := \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\}.$$

It is actually checked that the sum of components of $F_{\mathrm{AP}}(t)$ is 1 a.s. Then Bertoin [9] noticed (it was implicit in [3]) that the process $(F_{\mathrm{AP}}(t), t \geq 0)$ is an S-valued self-similar fragmentation with index $1/2$, in the following sense.

Definition 1 *An S-valued self-similar fragmentation with index $\beta \in \mathbb{R}$ is an S-valued Markov process starting a.s. from $(1, 0, \ldots)$, which is continuous in probability and satisfies the following* fragmentation property *:*

> *Given $F(t) = \mathbf{s} = (s_1, s_2, \ldots)$, the law of $F(t+t')$ is that of the decreasing rearrangement of the sequences $s_i F^{(i)}(s_i^{\beta} t'), i \geq 1$, where the $F^{(i)}$'s are independent copies of F.*

Such fragmentations have been introduced and extensively studied by Bertoin in [8, 9]. By [5], the laws of the self-similar fragmentations are characterized by a 3-tuple (β, c, ν), where β is the self-similarity index, $c \geq 0$ is an erosion coefficient and, more importantly, ν is a σ-finite *dislocation measure* on S that integrates the map $\mathbf{s} \mapsto 1 - s_1$. This measure ν describes the "jumps" of the fragmentation process, i.e. the way sudden dislocations occur. Roughly speaking, $x^{\beta}\nu(d\mathbf{s})$ is the instantaneous rate at which an object with size x fragments to form objects with sizes $x\mathbf{s}$ (see also Lemma 10 below). In [9], Bertoin showed

that the erosion of F_{AP} is 0, and that the dislocation measure ν_{AP} is characterized by the two formulas

$$\nu_{\mathrm{AP}}(s_1 \in \mathrm{d}x) = \frac{\mathrm{d}x}{\sqrt{2\pi x^3(1-x)^3}}\,, \qquad x \in [1/2, 1),$$

and $\nu_{\mathrm{AP}}\{\mathbf{s} : s_1 + s_2 < 1\} = 0$ (such fragmentations are called *binary*).

The main motivation of the present paper is to seek for a possible generalization of the fragmentation F_{AP}, when the Brownian CRT is replaced by the general $\alpha \in (1,2)$-stable tree. The game is made interesting in that there are important structural differences between the Brownian tree and the other stable trees, which imply that the Aldous-Pitman fragmentation device explained above (homogeneous fragmentation on the skeleton) gives rise to a binary fragmentation process which is *not* self-similar. It seems that the fragmentations hence obtained are related to the ones studied in [20] in relation with the additive coalescent, but this will be studied elsewhere. The defect in the self-similarity property comes from the fact that, contrary to the Brownian tree which is *binary* (its branchpoints have degree 3), the branchpoints of the stable tree are *hubs* with infinite degree and with different "magnitudes". These are not affected by the Aldous-Pitman fragmentation device, which a.s. never cuts at branchpoints. Therefore, as time passes, this device creates small trees with unusually "large" hubs, which cannot be rescaled copies of the initial stable tree. Rather, to obtain self-similarity, it is needed to directly remove the hubs themselves with a certain strategy.

Call $\mathcal{H}(\mathcal{T})$ the set of branchpoints of $\mathcal{T}$, which will also be referred to as the set of hubs of $\mathcal{T}$ when dealing with the stable ($\alpha \in (1,2)$) tree. To evaluate the magnitude of $b \in \mathcal{H}(\mathcal{T})$, consider the *fringe subtree* $\mathcal{T}_b$ rooted at b, i.e. the subset $\{v \in \mathcal{T} : b \in [[\mathrm{root}, v]]\}$. Then one can define the *local time*, or *width* of the hub b as the limit

$$L(b) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mu\{v \in \mathcal{T}_b : d(v,b) < \varepsilon\} \tag{1}$$

which exists a.s. and is positive : see Proposition 2 below.

Now given a realization of $\mathcal{T}$ and for every $b \in \mathcal{H}(\mathcal{T})$, take a standard exponential random variable e_b, so that the variables e_b are independent as b varies (notice that $\mathcal{H}(\mathcal{T})$ is countable). For all $t \geq 0$ define an equivalence relation $\sim_t$ on $\mathcal{T}$ by saying that $v \sim_t w$ if and only if the path $[[v,w]]$ does not contain any hub b for which $e_b < tL(b)$. Alternatively, following more closely the spirit of Aldous-Pitman's fragmentation, we can also say that we consider Poisson point process $(b(t), t \geq 0)$ on the set of hubs with intensity $\mathrm{d}t \otimes \sum_{b \in \mathcal{H}(\mathcal{T})} L(b)\delta_b(\mathrm{d}v)$, and for each t we let $v \sim_t w$ if and only if no atom of the Poisson process that has appeared before time t belongs to the path $[[v,w]]$. We let $\mathcal{T}_1^t, \mathcal{T}_2^t, \ldots$ be the distinct equivalence classes for $\sim_t$, ranked according to the decreasing order of their μ-masses (provided these are well-defined quantities). It is easy to see that these sets are trees (in the same sense as $\mathcal{T}$), and that the families $(\mathcal{T}_i^t, i \geq 1)$ are nested as t varies, that is, for every $t' > t$ and $i \geq 1$, there exists $j \geq 1$ such that $\mathcal{T}_i^{t'} \subset \mathcal{T}_j^t$. If we let $F^+(t) = (\mu(\mathcal{T}_1^t), \mu(\mathcal{T}_2^t), \ldots)$, F^+ is thus a fragmentation process in the sense that $F^+(t')$ is obtained by splitting at random the elements of $F^+(t)$. We mention that the fragmentation F^+ is also considered and studied in the work in preparation [1], with independent methods.

We now state our main result, postponing definitions and properties of stable subor-

dinators to the next section. Let

$$D_\alpha = \frac{\alpha(\alpha-1)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(2-\alpha)} = \frac{\alpha^2\Gamma\left(2-\frac{1}{\alpha}\right)}{\Gamma(2-\alpha)}.$$

Theorem 1 *The process F^+ is a self-similar fragmentation with index $1/\alpha \in (1/2, 1)$ and erosion coefficient $c = 0$. Its dislocation measure ν_α is characterized by*

$$\nu_\alpha(G) = D_\alpha E\left[T_1 G(T_1^{-1}\Delta T_{[0,1]})\right]$$

for any positive measurable function G, where $(T_x, 0 \le x \le 1)$ is a stable subordinator with index $1/\alpha$, characterized by the Laplace transform

$$E[\exp(-\lambda T_1)] = \exp(-\lambda^{1/\alpha}) \qquad \lambda \ge 0,$$

and $\Delta T_{[0,1]}$ is the sequence of the jumps of T, ranked by decreasing order of magnitude.

In a companion paper [21], we studied a self-similar fragmentation process $(F^-(t), t \ge 0)$ which consisted in the decreasing sequences of the μ-masses of the connected components of the set $\{v \in \mathcal{T} : \mathrm{ht}(v) > t\}$ at time t, i.e. the forest obtained by putting aside the vertices of the stable tree height less than t. This fragmentation was studied in the Brownian case by Bertoin [9], although this work does not mention trees and only uses the encoding height process, which is well-known to be twice the standard Brownian excursion, and it was showed that it was self-similar with characteristics $(-1/2, 0, \nu_{\mathrm{AP}})$ (in [9] the dislocation measure is found to be $2\nu_{\mathrm{AP}}$, but it is done with a different normalization, using the standard excursion instead of twice this excursion). In [21], we showed that F^- has characteristics $(1/\alpha - 1, 0, \nu_\alpha)$, with ν_α as in Theorem 1. Bertoin's observation that the two devices described above for fragmenting the Brownian CRT are "dual" (same dislocation measure but indices with different signs) is therefore quite surprisingly generalized in the larger context of stable trees. Heuristically, this is made possible by an exchangeability property of the root of the stable tree with other vertices (with respect to the measure μ), which indeed suggests that when removing a hub or removing the vertices below a given hub, the subsequent forests will have the same law up to rescaling.

Let us now present a second motivation for studying the fragmentation F^+. As the rest of the paper will show, our proofs involve a lot the theory of Lévy processes, and compared with the study of F^-, which made a consequent place to combinatoric tree structures, the study of F^+ will be mainly "analytic". The fact that Lévy processes may be involved in fragmentation processes is not new. According to [7] and [20], adding a drift to a certain class of Lévy processes allows to construct interesting fragmentations related to the entrance boundary of the stochastic additive coalescent. Here, rather than adding a drift, which by analogy between [4] and [7] amounts to cut the skeleton of a continuum random tree with a homogeneous Poisson process, we will perform a "removing the jumps" operation analog to our inhomogeneous cutting on the hubs of the tree.

Precisely, let $(X_s, s \ge 0)$ be the canonical process in the Skorokhod space $\mathbb{D}([0,\infty), \mathbb{R})$ and let P be the law of the stable Lévy process with index $\alpha \in (1, 2)$, upward jumps only, characterized by the Laplace exponent

$$E[\exp(-\lambda X_1)] = \exp(\lambda^\alpha).$$

As we will recall from the work of Chaumont [12] in the following section, we may define the law $N^{(1)}$ of the excursion with unit duration of this process above its infimum process. Under this law, $X_s = 0$ for $s > 1$, so we let $\Delta X_{[0,1]}$ be the sequence of the jumps $\Delta X_s = X_s - X_{s-}$ for $s \in (0,1]$, ranked in decreasing order of magnitude. Consider the following marking process on the jumps : conditionally on X, let $(e_s, s : \Delta X_s > 0)$ be a family of independent random variables with standard exponential distribution, indexed by the countable set of jump-times of X. For every $t \geq 0$ let

$$Z_s^{(t)} = \sum_{0 \leq u \leq s} \Delta X_u \mathbb{1}_{\{e_u < t \Delta X_u\}}.$$

That is, each jump with magnitude Δ is marked with probability $1 - \exp(-t\Delta)$ independently of the other jumps and consistently as t varies, and $Z^{(t)}$ is the process that sums the marked jumps. We will see that $Z^{(t)}$ is finite a.s., so we may define $X^{(t)} = X - Z^{(t)}$ under $N^{(1)}$. Let

$$\underline{X}_s^{(t)} = \inf_{0 \leq u \leq s} X_u^{(t)} \quad , \quad 0 \leq s \leq 1,$$

and let $F^\natural(t)$ be the sequence of lengths of the constancy intervals of the process $\underline{X}^{(t)}$, ranked in decreasing order.

Theorem 2 *The process $(F^\natural(t), t \geq 0)$ has the same law as $(F^+(t), t \geq 0)$.*

We organize the paper as follows. In Sect. 2 we recall some facts about Lévy processes, excursions, and conditioned subordinators that will be crucial for our study. In Sect. 3 we give the rigorous description of Duquesne and Le Gall's Lévy trees, and rephrase the definition of F^+ given above in terms of a partition of the unit interval associated to a certain marked excursion of a stable Lévy process. Sections 4 and 5 are then respectively dedicated to the study of F^+ and $F^\natural$. Asymptotic results are finally given concerning the behavior at small and large times of F^+ in Sect. 6.

2 Some facts about Lévy processes

2.1 Stable processes, inverse subordinators

Let $(X_s, s \geq 0)$ be the canonical process in the Skorokhod space $\mathbb{D}([0,\infty), \mathbb{R})$ of càdlàg paths on $[0,\infty)$. We fix $\alpha \in (1,2)$. Let P be the law on $\mathbb{D}([0,\infty), \mathbb{R})$ that makes X the spectrally positive stable process with index α, that is, X has independent and stationary increments under P, it has only positive jumps, and its marginal law at some (and then all) $s > 0$ has Laplace transform given by the Lévy-Khintchine formula :

$$E[e^{-\lambda X_s}] = \exp(s\lambda^\alpha) = \exp\left(s \int_0^\infty \frac{C_\alpha \mathrm{d}x}{x^{1+\alpha}} (e^{-\lambda x} - 1 + \lambda x) \right), \quad \lambda \geq 0, \tag{2}$$

where $C_\alpha = \alpha(\alpha - 1)/\Gamma(2-\alpha)$. A fundamental property of X under P is the *scaling property*

$$\left(\frac{1}{\lambda^{1/\alpha}} X_{\lambda s}, s \geq 0 \right) \stackrel{d}{=} (X_s, s \geq 0) \quad \text{for all } \lambda > 0.$$

We let $(p_s(x), s > 0, x \in \mathbb{R})$ be the density with respect to Lebesgue measure of the law $P(X_s \in \mathrm{d}x)$, which is known to exist and to be jointly continuous in s and x.

Denote by $\underline{X}$ the infimum process of X defined by

$$\underline{X}_s = \inf_{0 \le u \le s} X_u\,, \quad s \ge 0.$$

Let T be the right-continuous inverse of the increasing process $-\underline{X}$ defined by

$$T_x = \inf\{s \ge 0 : \underline{X}_s < -x\}.$$

Then it is known that under P, T is a stable subordinator with index $1/\alpha$, that is, an increasing Lévy process with Laplace exponent

$$E[e^{-\lambda T_x}] = \exp(-x\lambda^{1/\alpha}) = \exp\left(-x\int_0^\infty \frac{c_\alpha \mathrm{d}y}{y^{1+1/\alpha}}(1-e^{-\lambda y})\right) \quad \text{for} \quad \lambda, x \ge 0,$$

where $c_\alpha = (\alpha\Gamma(1-1/\alpha))^{-1}$. We denote by $(q_x(s), x, s > 0)$ the family of densities with respect to Lebesgue measure of the law $P(T_x \in \mathrm{d}s)$, by [6, Corollary VII.1.3] they are given by

$$q_x(s) = \frac{x}{s}p_s(-x). \tag{3}$$

We also introduce the notations P^s for the law of of the processes X under P, killed at time s, and $P^{(-x,\infty)} := P^{T_x}$ for the law of the process killed when it first hits $-x$.

Let us now discuss the conditioned forms of distributions of jumps of subordinators. An easy way to obtain regular versions for these conditional laws is developed in [23, 24]. First, we define the size-biased permutation of the sequence $\Delta T_{[0,x]}$ of the ranked jumps of T in the interval $[0, x]$ as follows. Write $\Delta T_{[0,x]} = (\Delta_1(x), \Delta_2(x), \ldots)$ with $\Delta_1(x) \ge \Delta_2(x) \ge \ldots$, and recall that $T_x = \sum_i \Delta_i(x)$. We define, following [23, 24], the size-biased ordered sequence $\Delta_k^*(x), k \ge 1$ as follows. Let 1^* be a r.v. such that

$$P(1^* = i|\Delta T_{[0,x]}) = \frac{\Delta_i(x)}{T_x}$$

for all $i \ge 1$, and set $\Delta_1^*(x) = \Delta_{1^*}(x)$. Recursively, let k^* be such that

$$P(k^* = i|\Delta T_{[0,x]}, (j^*, 1 \le j \le k-1)) = \frac{\Delta_i(x)}{T_x - \Delta_1^*(x) - \ldots - \Delta_{k-1}^*(x)}$$

for $i \ge 1$ distinct of the j^*, $1 \le j \le k-1$, and finally set $\Delta_k^*(x) = \Delta_{k^*}(x)$. Then

Lemma 1 (i) *For $k \ge 1$,*

$$P\left(\Delta_k^*(x) \in \mathrm{d}y \,\middle|\, T_x, (\Delta_j^*(x), 1 \le j \le k-1)\right) = \frac{c_\alpha x q_x(s-y)}{s y^{1/\alpha} q_x(s)}\mathrm{d}y$$

where $s = T_x - \Delta_1^*(x) - \ldots - \Delta_{k-1}^*(x)$.

(ii) *Consequently, given $T_x = t, \Delta_1^*(x) = y$, the sequence $(\Delta_2^*(x), \Delta_3^*(x), \ldots)$ has the same law as $(\Delta_1^*(x), \Delta_2^*(x) \ldots)$ given $T_x = t - y$. Conversely, if we are given a random variable Y with same law as $\Delta_1^*(x)$ given $T_x = t$ and, given $Y = y$, a sequence $(Y_1, Y_2, \ldots)$ with same law as $(\Delta_1^*(x), \Delta_2^*(x))$ given $T_x = t - y$, then $(Y, Y_1, Y_2, \ldots)$ has same law as $(\Delta_1^*(x), \Delta_2^*(x), \ldots)$ given $T_x = t$.*

This gives a regular conditional version for $(\Delta_i^*(x), i \ge 1)$ given T_x, and thus induces a conditional version for $\Delta T_{[0,x]}$ given T_x by ranking.

2.2 Marked processes

We are now going to enlarge the original probability space to mark the jumps of the stable process. We let M_X be the law of a sequence $\mathbf{e} = (e_s, s : \Delta X_s > 0)$ of independent standard exponential random variables, indexed by the (countable) set of times where the canonical process X jumps[1]. We let $\mathbf{P}(\mathrm{d}X, \mathrm{d}\mathbf{e}) = P(\mathrm{d}X) \otimes M_X(\mathrm{d}\mathbf{e})$. This probability allows to mark the jumps of X, precisely we say that a jump occurring at time s is marked at level $t \geq 0$ if $e_s < t\Delta X_s$. Write

$$Z_s^{(t)} = \sum_{0 \leq u \leq s} \Delta X_u \mathbb{1}_{\{e_u < t\Delta X_u)\}}$$

for the cumulative process of marked jumps at level t. We also let $X^{(t)} = X - Z^{(t)}$. We know that the process $(\Delta X_s, s \geq 0)$ of the jumps of X is under P a Poisson point process with intensity $C_\alpha x^{-1-\alpha}\mathrm{d}x$ on $(0, \infty)$, it is then standard that the process $(\Delta Z_s^{(t)}, s \geq 0)$ is a Poisson point process with intensity $C_\alpha x^{-\alpha-1}(1 - e^{-tx})\mathrm{d}x$, meaning that under $\mathbf{P}$, $Z^{(t)}$ is a subordinator with no drift and Lévy measure $C_\alpha x^{-\alpha-1}(1-e^{-tx})\mathrm{d}x$, more precisely its Laplace transforms are given by

$$\mathbf{E}[e^{-\lambda Z_s^{(t)}}] = \exp\left(-s \int_0^\infty C_\alpha (1 - e^{-tx}) \frac{1 - e^{-\lambda x}}{x^{\alpha+1}} \mathrm{d}x\right) = \exp(-s(\lambda + t)^\alpha + s\lambda^\alpha + st^\alpha).$$

We denote by $(\rho_s^{(t)}(x), s, x \geq 0)$ the densities of the laws $P(Z_s^{(t)} \in \mathrm{d}x)$. It can be checked by [25, Proposition 28.3] from the expression of the Lévy measure of $Z^{(t)}$ that these densities exist and are jointly continuous. Likewise, the process $X^{(t)}$ is under $\mathbf{P}$ a Lévy process with Lévy measure $C_\alpha e^{-tx}x^{-\alpha-1}\mathrm{d}x$, and the Laplace transform of $X_s^{(t)}$ is given by

$$\mathbf{E}[e^{-\lambda X_s^{(t)}}] = \exp\left(s\lambda\alpha t^{\alpha-1} + s\int_0^\infty C_\alpha e^{-tx} \frac{\mathrm{d}x}{x^{\alpha+1}} (e^{-\lambda x} - 1 + \lambda x)\right) = \exp(s(\lambda + t)^\alpha - st^\alpha),$$

which is obtained by dividing the Laplace exponent of X_s by that of $Z_s^{(t)}$.

We now state an absolute continuity result that is analogous to Cameron-Martin's formula for Brownian motion with drift.

Proposition 1 *For every $t, s \geq 0$, we have the following absolute continuity relation : for every positive measurable functional F,*

$$\mathbf{E}[F(X_u^{(t)}, 0 \leq u \leq s)] = E[\exp(-st^\alpha - tX_s)F(X_u, 0 \leq u \leq s)].$$

Proof. By the expression for the Laplace exponent of $X^{(t)}$, we get

$$\mathbf{E}[e^{-\lambda X_s^{(t)}}] = e^{-st^\alpha} E[e^{-(\lambda+t)X_s}],$$

hence giving $\mathbf{P}(X_s^{(t)} \in \mathrm{d}x) = e^{-st^\alpha - tx} P(X_s \in \mathrm{d}x)$. The result easily follows by the Markov property. $\square$

[1] One way to attach such variables in a measurable way to the ω-dependent set of times $\{s : \Delta X_s > 0\}$ is to consider a doubly-indexed family $(e_{i,j}, i, j \geq 1)$ of iid standard exponential variables independent of X, and to attach $e_{i,j}$ to the time of occurrence of the i-th largest jump of X in the interval $[j-1, j)$.

Remark. Such an identity is a special case of the so called *density transformations* for Lévy processes, see e.g. [25, Theorem 33.2].

As a first consequence, it immediately follows that $X^{(t)}$ also has jointly continuous densities under $\mathbf{P}$, which are given by

$$p_s^{(t)}(x) = \frac{\mathbf{P}(X_s^{(t)} \in \mathrm{d}x)}{\mathrm{d}x} = \exp(-st^\alpha - tx)p_s(x).$$

We let $\underline{X}^{(t)}$ be the infimum process of $X^{(t)}$ and $T^{(t)}$ the right-inverse process of $-\underline{X}^{(t)}$, defined as we did above define $\underline{X}$ and T.

It is easily obtained that for every $t \geq 0$, the process $(X, Z^{(t)})$ is again a Lévy process under the law $\mathbf{P}$. We will also denote by $\mathbf{P}^s$, $\mathbf{P}^{(-x,\infty)}$ the laws derived from P^s and $P^{(-x,\infty)}$ by marking the jumps with M_X ; $Z^{(t)}$ and $X^{(t)}$ are then defined as before.

2.3 Bridges, excursions

For $r \in \mathbb{R}$ and $s > 0$ we will denote by $P^s_{0\to r}$ the law of the *stable bridge from* 0 *to* r *with length* s, so the family $(P^s_{0\to r}, r \in \mathbb{R})$ forms a regular conditional version for $P^s(\cdot | X_s = r)$. By [17], a regular version (which is the one we will always consider) is obtained as the unique law on the Skorokhod space $\mathbb{D}([0,s],\mathbb{R})$ that satisfies the following absolute continuity relation : for every $a \in (0,s)$ and any continuous functional F,

$$P^s_{0\to r}(F(X_u, 0 \leq u \leq s-a)) = E\left[F(X_u, 0 \leq u \leq s-a)\frac{p_a(r - X_{s-a})}{p_s(r)}\right]. \tag{4}$$

We let $\mathbf{P}^s_{0\to r}$ be the marked analog of $P^s_{0\to r}$ on an enriched probability space. Notice that Proposition 1 immediately implies that the bridge laws for the process $X^{(t)}$ under $\mathbf{P}$ are the same as those of X. Stable bridges from 0 to 0 satisfy the following scaling property : under $P^v_{0\to 0}$, the process $(v^{-1/\alpha}X_{vs}, 0 \leq s \leq 1)$ has law $P^1_{0\to 0}$.

Lemma 2 *The following formula holds for any positive measurable f, g, H :*

$$\begin{aligned}
&E^1_{0\to 0}\left[H(X)\sum_{0\leq s\leq 1}\Delta X_s f(s) g(\Delta X_s)\right] \\
&\quad = \int_0^1 \mathrm{d}s\, f(s)\int_0^\infty \mathrm{d}x \frac{C_\alpha p_1(-x)}{x^\alpha p_1(0)} g(x) E^1_{0\to -x}[H(X\oplus(s,x))],
\end{aligned}$$

where $X \oplus (s,x)$ is the process X to which has been added a jump at time s with magnitude x. Otherwise said, a stable bridge from 0 to 0 together with a jump $(s, \Delta X_s)$ picked according to the σ-finite measure $m(\mathrm{d}s, \mathrm{d}x) = \sum_{u:\Delta X_u>0}\Delta X_u \delta_{(u,\Delta X_u)}(\mathrm{d}s,\mathrm{d}x)$ is obtained by taking a stable bridge from 0 to $-x$ and adding a jump with magnitude x at time s, where s is uniform in $(0,1)$ and x is independent with σ-finite "law" $C_\alpha p_1(-x)p_1(0)^{-1}x^{-\alpha}\mathrm{d}x$.

Proof. By the Lévy-Itô decomposition of Lévy processes, one can write, under P, that X_s is the compensated sum

$$X_s = \lim_{\varepsilon\to 0}\left(\sum_{0\leq u\leq s}\Delta X_u \mathbb{1}_{\{\Delta X_u > \varepsilon\}} - (\alpha-1)^{-1}C_\alpha \varepsilon^{1-\alpha}s\right), \quad s \geq 0,$$

where $(\Delta X_u, u \geq 0)$ is a Poisson point process with intensity $C_\alpha x^{-\alpha-1}\mathrm{d}x$, and where the convergence is almost sure. By the Palm formula for Poisson processes, we obtain that for positive measurable f, g, h, H :

$$\begin{aligned} & E^1\left[h(X_1)H(X)\sum_{0\leq s\leq 1}\Delta X_s f(s) g(\Delta X_s)\right] \\ &= \int_0^1 \mathrm{d}s\, f(s)\int_0^\infty \mathrm{d}x \frac{C_\alpha}{x^\alpha} g(x) E^1[h(x+X_1)H(X\oplus(s,x))]. \end{aligned}$$

The result is then obtained by disintegrating with respect to the law of X_1. $\square$

We now state a useful decomposition of the stable bridge from 0 to 0. Recall that $(\rho_s^{(t)}(x), x\geq 0)$ is the density of $Z_s^{(t)}$ under $\mathbf{P}$ and that $X_1^{(t)} + Z_1^{(t)} = X_1$, which is a sum of two independent variables. From this we conclude that $(p_1(0)^{-1}p_1^{(t)}(x)\rho_1^{(t)}(x), x\geq 0)$ is a probability density on $\mathbb{R}_+$.

Lemma 3 *Take a random variable $\mathcal{Z}$ with law $P(\mathcal{Z}\in \mathrm{d}z) = p_1^{(t)}(-z)\rho_1^{(t)}(z)p_1(0)^{-1}\mathrm{d}z$. Conditionally on $\mathcal{Z}=z$, take X' with law $P^1_{0\to -z}$ and Z with law $\mathbf{P}^1(Z^{(t)}\in\cdot|Z_1^{(t)}=z)$, independently. That is, Z is the bridge of $Z^{(t)}$ with length 1 from 0 to z. Then $X'+Z$ has law $P^1_{0\to 0}$.*

Remark. The definition for the bridges of $Z^{(t)}$ under $\mathbf{P}^1$ has not been given before. One can either follow an analogous definition as (4), or use Lemma 1 about conditioned jumps of subordinators. We explain this for bridges of T, the construction for bridges of $Z^{(t)}$ being similar. Take $(\Delta_i, i\geq 1)$ a sequence whose law is that of the jumps $\Delta T_{[0,1]}$ of T under P before time 1, ranked in decreasing order, and conditioned by $T_1 = z$, in the sense of Lemma 1. Take also a sequence $(U_i, i\geq 1)$ of independent uniformly distributed random variables on $[0,1]$, independent of $\Delta T_{[0,1]}$. Then one checks from the Lévy-Itô decomposition for Lévy processes that the law Q_z of the process $T_s^{\mathrm{br}} = \sum \Delta_i \mathbb{1}_{\{s\geq U_i\}}$, with $0\leq s\leq 1$, defines as z varies a regular version of the conditional law $P^1(T\in\cdot|T_1=z)$.

Proof. Recall that under $\mathbf{P}^1$, X can be written as $X^{(t)}+Z^{(t)}$ with $X^{(t)}$ and $Z^{(t)}$ independent. Consequently, for f and G positive continuous, we have

$$E^1[f(X_1)G(X)] = \mathbf{E}^1[f(X_1^{(t)}+Z_1^{(t)})G(X^{(t)}+Z^{(t)})]$$

so

$$\begin{aligned} \int_{\mathbb{R}} \mathrm{d}x\, p_1(x) f(x) E^1_{0\to x}[G(X)] &= \int_{\mathbb{R}} \mathrm{d}x\, p_1(x)\int_0^\infty \mathrm{d}z \frac{p_1^{(t)}(x-z)\rho_1^{(t)}(z)}{p_1(x)} f(z) \\ &\quad \times \mathbf{E}^1[G(X^{(t)}+Z^{(t)})|X_1^{(t)} = x-z, Z_1^{(t)} = z]. \end{aligned}$$

Thus, for (Lebesgue) almost every x, the bridge with law $P^1_{0\to x}$ is obtained by taking a bridge of $X^{(t)}$ (or X by previous remarks) from 0 to $-\mathcal{Z}_x$ and an independent bridge of $Z^{(t)}$ from 0 to $\mathcal{Z}_x$, where $\mathcal{Z}_x$ is a r.v. with law $\mathrm{d}z\, p_1(x)^{-1}p_1^{(t)}(x-z)\rho_1^{(t)}(z)$ on $\mathbb{R}_+$. We extend this result to every $x\in\mathbb{R}$ by an easily checked continuity result for the laws of bridges which stems from (4) and the continuity of the densities. Taking $x=0$ gives the result. $\square$

We now turn our attention to excursions. The fact that X has no negative jumps implies that $-\underline{X}$ is a local time at 0 for the reflected process $X-\underline{X}$. Let N be the Itô

excursion measure of $X-\underline{X}$ away from 0, so that the path of $X-\underline{X}$ is obtained by concatenation of the atoms of a Poisson measure with intensity $N(\mathrm{d}X)\otimes\mathrm{d}t$ on $\mathbb{D}^\dagger([0,\infty),\mathbb{R})\times\mathbb{R}_+$, where $\mathbb{D}^\dagger([0,\infty),\mathbb{R})$ denotes the Skorokhod space of paths that are killed at some time ζ. Under N, almost every path X starts at 0, is positive on an interval $(0,\zeta)$ and dies at the first time $\zeta(X)\in(0,\infty)$ it hits 0 again. We let $\mathbf{N}$ be the enriched law with marked jumps. It follows from excursion theory that the Lévy process $(X,Z^{(t)})$ under $\mathbf{P}$ is obtained by taking a Poisson point measure $\sum_{i\in I}\delta_{X^i,\mathbf{e}^i,s^i}$ indexed by a countable set I, with intensity $\mathbf{N}(\mathrm{d}X,\mathrm{d}\mathbf{e})\otimes\mathrm{d}s$, writing $Z^{(t),i}$ for the cumulative process of marked jumps for X^i and letting

$$X_s=-s^i+X^i\left(s-\sum_{j:s^j<s^i}\zeta_j(X^j)\right)$$

and

$$Z^{(t)}_s=\sum_{j:s^j<s^i}Z^{(t),j}_{\zeta_j(X^j)}+Z^{(t),i}\left(s-\sum_{j:s^j<s^i}\zeta_j(X^j)\right),$$

whenever $\sum_{j:s^j<s^i}\zeta_j(X^j)\le s\le\sum_{j:s^j\le s^i}\zeta_j(X^j)$.

If X is stopped at some time s, for any $u\in[0,s]$ we define the rotated process

$$V_uX(r)=(X_{r+u}-X_u)\mathbb{1}_{\{0\le u<s-u\}}+(X(r-s+u)+X_s-X_u)\mathbb{1}_{\{s-u\le r\le s\}}.$$

Let $m_s=-\underline{X}_s$ and suppose that this minimum is attained only once on $[0,s]$. We define the Vervaat transform of X as $VX=V_{T(m_s-)}X$, the rotation of X at the time where it attains its infimum. Provided that $X_0=0$ and $X_s=X_{s-}=0$ (say that X is a bridge), VX is then an excursion-like function, starting and ending at 0, and staying positive in the meanwhile.

We will denote by $N^{(v)}$ the law of VX under $P^v_{0\to0}$, and $\mathbf{N}^{(v)}$ the corresponding "marked" version. Call it the law of the *excursion of X with duration v*. The "Vervaat theorem" in [12] shows that $N^{(v)}$ is indeed a regular conditional version for the "law" $N(\cdot|\zeta=v)$: for any positive measurable functional F and function f,

$$N(F(X_s,0\le s\le\zeta)f(\zeta))=\int_{\mathbb{R}_+}f(\zeta)N(\zeta\in\mathrm{d}v)N^{(v)}(F(X_s,0\le s\le v)).$$

As for bridges, we also have the scaling property at the level of conditioned excursions : under $N^{(v)}$, $\left(v^{-1/\alpha}X_{vs},0\le s\le1\right)$ has law $N^{(1)}$. Notice also (either by Vervaat's theorem or directly, using Proposition 1) that the excursions of $X^{(t)}$ under $\mathbf{P}$, conditioned to have a fixed duration v are the same as that of X under $N^{(v)}$.

3 The stable tree

3.1 Height Process, width process

We now introduce the rigorous definition and useful properties of the stable tree. This section is mainly inspired by [16, 14]. With the notations of section 2, and for $t\ge0$, let $R^{(t)}$ be the time-reversed process of X at time t :

$$R^{(t)}_s=X_t-X_{(t-s)-}\quad 0\le s\le t.$$

It is standard that this process has the same law as X killed at time t under P. Let $\overline{R}^{(t)}$ be its supremum process, and $\widehat{L}^{(t)}$ be the local time process at level 0 of the reflected process $\overline{R}^{(t)} - R^{(t)}$. We let $H_t = \widehat{L}^{(t)}_t$. The normalization for $\widehat{L}^{(t)}$ is chosen so that

$$H_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}\{\overline{R}^{(t)}_s - R^{(t)}_s \leq \varepsilon\} \mathrm{d}s,$$

in probability for every t. It is proved in [16] that H admits a continuous modification, which is the one we are going to work with from now on. It has to be noticed that H_t is not a Markov process, except in the case where X is Brownian motion. As a matter of fact, it can be noticed that H admits infinitely many local minima attaining the same value as soon as X has jumps. To see this, consider a jump time t of X, and let $t_1, t_2 > t$ so that $\inf_{t \leq u \leq t_i} X_u = X_{t_i}$ and $X_{t-} < X_{t_i} < X_t$, $i \in \{1, 2\}$. Then it is easy to see that $H_t = H_{t_1} = H_{t_2}$ and that one may in fact find an infinite number of distinct t_i's satisfying the properties of t_1, t_2. On the other hand, it is not difficult to see that H_t is a local minimum of H, see Proposition 2 below.

It is shown in [16] that the definition of H still makes sense under the σ-finite measure N rather than the probability law P. The process H is then defined only on $[0, \zeta]$, and we call it the *excursion of the height process*. Using the scaling property, one can then define the height process under the laws $N^{(v)}$. Call it the law of the excursion of the height process with duration v.

The key tool for defining the local time of hubs is the local time process of the height process. We will denote by $(L^t_s, t, s \geq 0)$. It can be obtained a.s. for every fixed s, t by

$$L^t_s = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s \mathbb{1}_{\{t < H_u \leq t + \varepsilon\}} \mathrm{d}u. \tag{5}$$

That is, L^t_s is the density of the occupation measure of H at level t and time s. For $t = 0$, one gets $(L^0_s, s \geq 0) = (\underline{X}_s, s \geq 0)$, which is a reminiscent of the fact that the excursions of the height process are in one-to-one correspondence with excursions of X with the same lengths.

It is again possible to define the local time process under the excursion measures N and $N^{(v)}$. Duquesne and Le Gall [16] have shown that under P, the process $(L^t_{T_x}, t \geq 0)$ has the law of the continuous-stable branching process starting at $x > 0$, with stable (α) branching mechanism. One can get interpretations for the process $(L^t_\zeta, t \geq 0)$ under the measure N or of $(L^t_v, t \geq 0)$ under $N^{(v)}$ in terms of conditioned continuous-state branching processes, see [21].

3.2 The tree structure

Let us motivate the term of "height process" for H by embedding a tree inside H, following [19, 2]. Consider the height process H under the law $N^{(1)}$. We can define a pseudo metric D on $[0, 1]$ by letting $D(s, s') = H_s + H_{s'} - 2 \inf_{u \in [s, s']} H_u$ (with the convention that $[s, s'] = [s', s]$ if $s' < s$). Let $s \equiv s'$ if and only if $D(s, s') = 0$.

Definition 2 *The* stable tree $(\mathcal{T}, d)$ *is the quotient of the pseudo-metric space* $([0, 1], D)$ *by* $\equiv$. *The* root *of* $\mathcal{T}$ *is the equivalence class of* 0. *The* mass measure μ *is the Borel measure induced on* $\mathcal{T}$ *by Lebesgue's measure on* $[0, 1]$ *(so its support is* $\mathcal{T}$*).*

In the sequel, we will often identify $\mathcal{T}$ with $[0,1]$, even if the correspondence is not one-to-one. Some comments on this definition. First, the way the tree is embedded in the function H can seem quite intricate. It is not difficult, however, to see what its "marginals" look like. For any finite set of vertices $s_1, s_2, \ldots, s_k \in [0,1]$, one recovers the structure of the subtree spanned by the root and $s_1, s_2, \ldots, s_k$, according to the following simple rules :

– The height of s is $\mathrm{ht}(s) = H_s$.
– The common ancestor of $s_1, \ldots, s_k$ is $b = b(s_1, \ldots, s_k) \in [\min_{1\le i\le k} s_i, \max_{1\le i\le k} s_i]$ such that $H_b = \inf\{H_s : s \in [\min_{1\le i\le k} s_i, \max_{1\le i\le k} s_i]\}$.

Notice that all such b are equivalent with respect to $\equiv$. The fact that $(\mathcal{T}, d)$ is indeed a tree (a complete metric space such that the only simple path leading from a vertex to another is the geodesic) is intuitive and proven in [15]. It follows from the construction of "marginals" of $\mathcal{T}$ in [16] that given μ, μ-a.e. vertex is a leaf of $\mathcal{T}$.

We now relate properties on the stable tree to path properties of the underlying Lévy process we started with to construct the height process. We understand here that X and H are defined under $N^{(1)}$. Recall that $\mathcal{T}_b$ stands for the fringe subtree rooted at b.

Proposition 2 (i) *Each hub $b \in \mathcal{H}(\mathcal{T})$ is encoded by exactly one time $\tau(b) \in [0,1]$ such that $L(b) = \Delta X_{\tau(b)} > 0$, and $L(b)$ is given by (1) a.s.*

(ii) *If $\sigma(b) = \inf\{s \ge \tau(b) : X_s = X_{\tau(b)-}\}$, then $\mathcal{T}_b = [\tau(b), \sigma(b)]/\equiv$.*

(iii) *More precisely, let $\mathcal{T}_b^1, \mathcal{T}_b^2, \ldots$ be the connected components of $\mathcal{T}_b \setminus \{b\}$, arranged in decreasing order of mass. Let $([\tau_i(b), \sigma_i(b)], i \ge 1)$ be the constancy intervals of the infimum process of $(X_s - X_{\tau(b)}, \tau(b) \le s \le \sigma(b))$, and ranked in decreasing order of length. Then $\mathcal{T}_b^i = (\tau_i(b), \sigma_i(b))/\equiv$.*

Proof. (i) Working first under P, fix $\ell > 0$ and let $\tau_\ell = \inf\{s \ge 0 : \Delta X_s > \ell\}$. Then τ_ℓ is a stopping time for the natural filtration associated to X, as well as $\sigma_\ell = \inf\{s > \tau_\ell : X_s = X_{\tau_\ell -}\}$. By the Markov property, the process $X_{[\tau_\ell, \sigma_\ell]} = (X_{\tau_\ell + s} - X_{\tau_\ell}, 0 \le s \le \sigma_\ell - \tau_\ell)$ is independent of $(X_{s+\sigma_\ell} - X_{\sigma_\ell}, s \ge 0)$, which has the same law as X, and of $(X_s, 0 \le s \le \tau_\ell)$ conditionally on its final jump ΔX_{τ_ℓ}. Now if we remove this jump, that is, if we let $(\widetilde{X}_s, 0 \le s \le \tau_\ell)$ be the modification of $(X_s, 0 \le s \le \tau_\ell)$ that is left-continuous at τ_ℓ, then $\widetilde{X}$ has the law of a stable Lévy process killed at some independent exponential time, and conditioned to have jumps with magnitude less than ℓ. Also, conditionally on $\Delta X_{\tau_\ell} = x$, $X_{[\tau_\ell, \sigma_\ell]}$ has the law $P^{(-x,\infty)}$ of the stable process killed when it first hits $-x$. Hence, by the additivity of the local time and the definition of H, one has that for every $s \in [\tau_\ell, \sigma_\ell]$, $H_s = H_{\tau_\ell} + \widetilde{H}_{s-\tau_\ell}$, where $\widetilde{H}$ is an independent copy of H, killed when its local time at 0 attains x. Consequently, one has $H_s \ge H_{\tau_\ell}$ for every $s \in [\tau_\ell, \sigma_\ell]$ and $H_{\sigma_\ell} = H_{\tau_\ell}$, moreover, one has that for every $\varepsilon > 0$,

$$\inf_{(\tau_\ell - \varepsilon)\vee 0 \le s \le \tau_\ell} H_s \vee \inf_{\sigma_\ell \le s \le \sigma_\ell + \varepsilon} H_s < H_{\tau_\ell}, \tag{6}$$

as a consequence of the following fact. By the left-continuity of X at τ_ℓ, for any $\varepsilon > 0$ we may find $s \in [\tau_\ell - \varepsilon, \tau_\ell]$ such that $\inf_{u \in [s, \tau_\ell]} X_u = X_s$. This implies $H_s = H_{\tau_\ell} - \widehat{L}^{(\tau_\ell)}_{\tau_\ell - s}$, and this last term is a.s. strictly less than H_{τ_ℓ} because 0 is is a.s. not a holding point for $(\widehat{L}^{(\tau_\ell)}_s, 0 \le s \le \tau_\ell)$. This last fact is obtained by a time-reversal argument, using the fact that the points of increase of the local time $\widehat{L}^{(t)}$ correspond to that of the supremum process of $R^{(t)}$. Moreover, the fact that X has only positive jumps under P implies that for some suitable $\varepsilon' > 0$, one can find some $s' \in [\sigma_\ell, \sigma_\ell + \varepsilon']$ and some $s'' \in [\tau_\ell - \varepsilon, \tau_\ell]$

such that $H_u \geq H_{s'} = H_{s''}$ for every $u \in [s', s'']$, and such that again $\inf_{u\in[s'',\tau_\ell]} X_u = X_{s''}$. Thus the claimed inequality. In terms of the structure of the stable tree, (6) implies that a branchpoint b of the tree is present at height H_{τ_ℓ}, which is encoded by all the $s \in [\tau_\ell, \sigma_\ell]$ such that $H_s = H_{\tau_\ell}$, i.e. such that $X_s = \inf_{u\in[\tau_\ell,s]} X_u$ (there is always an infinite number of them). By definition, the mass measure of the vertices in $\mathcal{T}_b$ at distance less than ε of b is exactly the Lebesgue measure of $\{s \in [\tau_\ell, \sigma_\ell] : \widetilde{H}_{s-\tau_\ell} < \varepsilon\}$. Thus by (5) we can conclude that $L(b)$ defined at (1) exists and equals $\widetilde{L}^0_{\sigma_\ell - \tau_\ell} = x$ where $\widetilde{L}$ is the local time associated to $\widetilde{H}$. The same argument allows to handle the second, third, ... jumps that are $> \ell$. Letting $\ell \downarrow 0$ implies that to any jump of X with magnitude x corresponds a hub of the stable tree with local time x. By excursion theory and scaling, the same property holds under N and $N^{(1)}$.

Conversely, suppose that b is a branchpoint in the stable tree. This means that there exist times $s_1 < s_2 < s_3$ such that $H_{s_1} = H_{s_2} = H_{s_3}$ and $H_s \geq H_{s_1}$ for every $s \in [s_1, s_3]$. Let

$$\tau(b) = \inf\{s \leq s_2 : H_s = H_{s_2} \text{ and } H_u \geq H_{s_2} \forall u \in [s, s_2]\}$$

and

$$\sigma(b) = \sup\{s \geq s_2 : H_s = H_{s_2} \text{ and } H_u \geq H_{s_2} \forall u \in [s_2, s]\}$$

(which are not stopping times). If $\Delta X_{\tau(b)} > 0$, we are in the preceding case. Suppose that $\Delta X_{\tau(b)} = 0$, then by the same arguments as above, $X_s \geq X_{\tau(b)}$ for $s \in [\tau(b), \sigma(b)]$, else we could find some $s' \in [\tau(b), \sigma(b)]$ such that $H_{s'} < H_{\tau(b)}$. Also, the points $s \in [\tau(b), \sigma(b)]$ such that $H_s = H_{\tau(b)}$ must then satisfy $X_s = X_{\tau(b)}$ (else there would be a strict increase of the local time of the reversed process). This implies that $X_{\tau(b)}$ is a local infimum of X, attained at s. By standard considerations, such local infima cannot be attained more than three times on the interval $[\tau(b), \sigma(b)]$, a.s. But if it was attained exactly three times, then the branchpoint would have degree 3, which is impossible according to the analysis of F^- in [21], which implies that all hubs of the stable tree have infinite degree.

Assertion (ii) follows easily from this, and (iii) comes from the fact that the points $u \in [\tau(b), \sigma(b)]$ with $H_u = H_{\tau(b)}$ are exactly those points where $\inf_{r\in[\tau(b),u]} X_r = X_u$, and the definition of the mass measure on $\mathcal{T}$. □

3.3 A second way to define F^+

We will now give some elementary properties of F^+ and rephrase its definition directly from the excursion of the underlying stable excursion X rather than the tree itself. First recall that given $\mathcal{T}$, we defined F^+ through a marking procedure on $\mathcal{H}(\mathcal{T})$ by taking a Poisson process $(b(t), t \geq 0)$ with intensity $\mathrm{d}t \otimes \sum_{b\in\mathcal{H}(\mathcal{T})} L(b)\delta_b(\mathrm{d}v)$, and by saying that b is marked at level t if $b \in \{b(s), 0 \leq s \leq t\}$. By proposition 2, F^+ can thus be defined under the marked law $\mathbf{N}^{(1)}$. To describe this construction a bit more, we begin with the following

Lemma 4 *Let $s \in [0, 1]$, and write $v(s)$ for the vertex of $\mathcal{T}$ encoded by s. Then almost-surely,*

$$\sum_{b\in\mathcal{H}(\mathcal{T})\cap[[\mathrm{root},v(s)]]} L(b) < \infty.$$

In particular, almost surely, for every hub $b \in \mathcal{H}(\mathcal{T})$ and $t \geq 0$, there is at most a finite number of hubs marked at level t on the path $[[\mathrm{root}, b]]$.

Proof. Let s be the leftmost time in $[0,1]$ that encodes v. It follows from Proposition 2 (ii) (and the fact that a.s. under P, every excursion of $R^{(s)}$ below $\overline{R}^{(s)}$ ends by a jump) that the hubs b in the path $[[\mathrm{root}, v]]$ are all encoded by the times $s' < s$ such that $\overline{R}^{(s)}$ jumps at time $s-s'$. This jump corresponds to a jump of the reversed process $R^{(s)}$, whose magnitude $\Delta R^{(s)}_{s-s'} \geq \Delta \overline{R}^{(s)}_{s-s'}$ equals $L(b)$ by Proposition 2 (i). Therefore, we have to show that the sum of these jumps is finite a.s. By excursion theory and time-reversal, it suffices to show that under P, letting $\overline{X}$ be the supremum process of X,

$$\sum_{0\leq s'\leq s : \Delta \overline{X}_{s'}>0} \Delta X_{s'} < \infty\,, \qquad s \geq 0. \tag{7}$$

Now by excursions and Poisson processes theories (see e.g. Formula (10) in the proof of [16, Lemma 1.1.2]), after appropriate time-change by the inverse local time at 0 of the process $X - \overline{X}$, the jumps $\Delta X_{s'}$ that achieve new suprema form a Poisson point process with intensity $x \times C_\alpha x^{-1-\alpha} \mathrm{d}x$. Since this measure integrates x on a neighborhood of 0, the sum in (7) is a.s. finite.

The statement on hubs follows since for any hub b encoded by a jump-time $\tau(b)$, there is a rational number $r' \in [\tau(b), \sigma(b)]$ which encodes some vertex v in the fringe subtree rooted at b. Therefore, almost-surely, for every $b \in \mathcal{H}(\mathcal{T})$, the sum of widths of the hubs on the path $[[\varnothing, b]]$ is finite. It is then easy to check that if $(x_1, x_2, \ldots)$ is a sequence with finite sum and if the i-th term is marked with probability $1 - e^{-tx_i}$, then a.s. only a finite number of terms are marked. Therefore, a.s. for every $b \in \mathcal{H}(\mathcal{T})$, there is only a finite number of marked hubs on the path $[[\varnothing, b]]$. $\square$

By definition, two vertices $v, w \in \mathcal{T}$ satisfy $v \sim_t w$ if and only if $\{b(s) : 0 \leq s \leq t\} \cap [[v, w]] = \emptyset$. Let $\mathcal{H}_t = \{b(s) : 0 \leq s \leq t\}$. For $b \in \mathcal{H}_t$, let $\mathcal{T}_b^1, \mathcal{T}_b^1, \ldots$ be the connected components of $\mathcal{T}_b \setminus \{b\}$ ranked in decreasing order of total mass. We know that these trees are encoded by intervals of the form $(\tau_i(b), \sigma_i(b))$ whose union is $[\tau(b), \sigma(b)] \setminus \{u : u \equiv b\}$. Define

$$C(t, b, i) = \mathcal{T}_b^i \setminus \bigcup_{b' \in \mathcal{H}_t \cap \mathcal{T}_b^i} \mathcal{T}_{b'},$$

so $C(t, b, i)$ is the connected component of the i-th largest subtree growing from b obtained when the hubs marked at level t are deleted. Plainly, $C(t, b, i)$ is an equivalence class for $\sim_t$ for every $b \in \mathcal{H}_t$ and $i \geq 1$. By (iii) in Proposition 2, with obvious notations,

$$C(t, b, i) \equiv (\tau_i(b), \sigma_i(b)) \setminus \bigcup_{b' \in \mathcal{T}_i^b \cap \mathcal{H}_t} [\tau(b'), \sigma(b')].$$

We also let $C(t, \emptyset)$ be the set of vertices whose path to the root does not cross any marked hub at level t, which is equivalent to $[0,1] \setminus \bigcup_{b \in \mathcal{H}_t} [\tau(b), \sigma(b)]$. Then $C(t, \emptyset)$ is also an equivalence class for $\sim_t$. Intuitively, the classes $C(t, \emptyset)$ and $C(t, b, i)$ for b a hub are the equivalence classes for $\sim_t$ that have a positive weight. We will see later that the rest is a set of leaves of mass zero.

Let us now translate the relation $\sim_t$ in terms of the stable excursion X under $\mathbf{N}^{(1)}$. Let $s, s' \in [0,1]$ encode respectively the vertices $v \neq w \in \mathcal{T}$. Again by Proposition 2 (ii), the branchpoint $b(v, w)$ of v and w is encoded by the largest u such that the processes $(\overline{R}^{(s)}_{s-u+r}, 0 \leq r \leq u)$ and $(\overline{R}^{(s')}_{s'-u+r}, 0 \leq r \leq u)$ coincide. Let $u(s, s')$ be the jump-time of X

that encodes this branchpoint. Then $v \sim_t w$ if and only if the (left-continuous) processes $(\overline{R}^{(s)}_{s-r}, u(v,w) \le r \le s)$ and $(\overline{R}^{(s)}_{s'-r}, u(v,w) \le r \le s')$ never jump at times when marked jumps at level t for X occur.

In particular, we may rewrite the equivalence classes $C(t,b,i)$ and $C(t,\emptyset)$ as follows. Let $z_1^t \ge z_2^t \ge \ldots \ge 0$ be the marked jumps of X at level t under $\mathbf{N}^{(1)}$, ranked in decreasing order, and let $\tau_1^t, \tau_2^t, \ldots$ the corresponding jump times (i.e. such that $\Delta Z^{(t)}_{\tau_i^t} = z_i^t$). For every i, let

$$\sigma_i^t = \inf\{s > \tau_i^t : X_s = X_{\tau_i^t -} = X_{\tau_i^t} - z_i^t\}$$

be the first return time to level $X_{\tau_i^t -}$ after time τ_i^t. Define the intervals $I_i^t = [\tau_i^t, \sigma_i^t]$, so $I_i^t / \equiv$ is the fringe subtree of the marked hub that has width z_i^t. Notice that the I_i^t's are by no means disjoint, since these fringe subtrees contain other marked hubs, that might even have greater width. For each i, the jump with magnitude z_i^t gives rise to a family of excursions of X above its minimum. Precisely, let $(X_{i,1}^t, X_{i,2}^t, \ldots)$ the sequence of excursions above its infimum of the process

$$X_i^t(s) = X_{\tau_i^t + s} - X_{\tau_i^t} \quad 0 \le s \le \sigma_i^t - \tau_i^t, i \ge 1$$

where the $(X_{i,j}^t, j \ge 1)$ are arranged by decreasing order of duration. Let also $I_{i,j}^t = [\tau_{i,j}^t, \sigma_{i,j}^t]$ be the interval in which $X_{i,j}^t$ appears in X, so that $\overline{\bigcup_j I_{i,j}^t} = I_i^t$. Consider the set

$$C_{i,j}^t = I_{i,j}^t \setminus \bigcup_{k : I_k^t \subsetneq I_i^t} I_k^t.$$

By Lemma 4, there exists some set of indices k' such that $I_{k'}^t \subsetneq I_{i,j}^t$ and so that the $I_{k'}^t$'s are maximal with this property (else we could find an infinite number of marked hubs on a path from the root to one of the hubs encoded by the left-end of some $I_k^t \subsetneq I_{i,j}^t$). The Lebesgue measure of $C_{i,j}^t$ is thus equal to

$$|C_{i,j}^t| = \sigma_{i,j}^t - \tau_{i,j}^t - \sum (\sigma_k^t - \tau_k^t),$$

where the sum is over the k's such that $I_k^t \subsetneq I_i^t$ and the I_k^t's are maximal with this property. Writing $C_0^t = [0,1] \setminus \bigcup_{i=1}^\infty I_i^t$, we finally get (identifying Borel subsets of $[0,1]$ with Borel subsets of $\mathcal{T}$) :

Lemma 5 *The sets C_0^t and $C_{i,j}^t$, for $i,j \ge 1$, are a relabeling of the sets $C(t,\emptyset)$ and $C(t,b,i)$.*

Notice also that another consequence of Lemma 4 is that F^+ is continuous in probability at time 0. Indeed, as $t \downarrow 0$, the component $C(t,\emptyset)$ of the fragmented tree containing the root increases to $C(0+,\emptyset)$. Suppose $\mu(C(0+,\emptyset)) < 1$ with positive probability. Given $\mathcal{T}$ take $L_1, L_2, \ldots$ independent with law μ. By the law of large numbers, with positive probability a positive proportion of the L_i's are separated from the root at time $0+$. However, as a consequence of Lemma 4, a.s. for every $n \ge 1$ and t small enough, there is no marked hub on the paths $[[\mathrm{root}, L_i]], 1 \le i \le n$, hence a contradiction.

4 Study of F^+

The goal of this section is to study the fragmentation F^+ through the representation given in the last section. The first step is to study the behavior of the excursion on the equivalence classes $C_{i,j}^t$ and C_0^t defined previously.

4.1 Self-similarity

This section is devoted to the proof that F^+ is a self-similar fragmentation with index $1/\alpha$ and no erosion.

Let us first introduce some notation. Let $(f(x), 0 \leq x \leq \zeta) \in \mathbb{D}^\dagger([0,\infty),\mathbb{R})$ be a càdlàg function with lifetime $\zeta \in [0,\infty)$. By convention we let $f(x) = f(\zeta)$ for $x > \zeta$. We define the unplugging operation `UNPLUG` as follows. Let $([a_n, b_n], n \geq 1)$ be a sequence of disjoint closed intervals with non-empty interior, such that $0 < a_n < b_n < \zeta$ for every n. Define the increasing continuous function

$$x^{-1}(s) = s - \sum_{n\geq 1}(s \wedge b_n - a_n)^+ , \qquad s \geq 0,$$

where $a^+ = a \vee 0$ and where the sum converges uniformly on $[0,\zeta]$. We say that the intervals $[a_n, b_n]$ are *separated* if $x^{-1}(a_n) < x^{-1}(a_m)$ for every $n \neq m$ such that $a_n < a_m$. This is equivalent to the fact that for every $n \neq m$ with $a_n < a_m$, the set $[a_n, a_m] \setminus \bigcup_i [a_i, b_i]$ has positive Lebesgue measure, and it implies that the constancy intervals of x^{-1} are exactly $[a_n, b_n], n \geq 1$. If $([a_n, b_n], n \geq 1)$ is separated, define x as the right-continuous inverse of x^{-1}, then $f \circ x$ is càdlàg (notice that $(f \circ x)(s-) = f(x(s-)-)$ for $s \in [0, x^{-1}(\zeta)]$), call it $\mathtt{UNPLUG}(f, [a_n, b_n], n \geq 1)$. The action of `UNPLUG` is thus to remove the bits of the path of f that are included in $[a_n, b_n]$. Last, if we are given intervals $[a_n, b_n]$ that are not overlapping (i.e. such that $a_n < a_m < b_n < b_m$ does not happen for $n \neq m$, though we might have $[a_n, b_n] \subset [a_m, b_m]$), but such that there is a separated subsequence $([a_{\phi(n)}, b_{\phi(n)}], n \geq 1)$ of maximal intervals that covers $\bigcup_n [a_n, b_n]$, we similarly define the unplugging operation by simply ignoring the non-maximal intervals.

Lemma 6 *Let $([a_n, b_n], n \geq 1)$ be a sequence of separated intervals, and let π be a partition of $\mathbb{N}$ with blocks $\pi_1, \pi_2, \ldots$. Then, as $N \to \infty$, $\mathtt{UNPLUG}(f, [a_n, b_n] : n \in \pi_1 \cup \ldots \cup \pi_N)$ converges to $\mathtt{UNPLUG}(f, [a_n, b_n], n \geq 1)$ in the Skorokhod topology.*

Proof. Define

$$x_N^{-1}(s) = s - \sum_{n \in \pi_1 \cup \ldots \cup \pi_N} (s \wedge b_n - a_n)^+ , \qquad s \geq 0.$$

The separation of intervals ensures that every jump of x corresponds to a jump of x_N for some large N, and it is not hard to see that this implies $x_N(x_N^{-1}(x(s))) = x(s)$ for all s. Since $f \circ x$ is càdlàg with duration $\zeta' = \zeta - \sum_n (b_n - a_n)$, for every N we may find a sequence of times $0 = s_0 < s_1 < s_2 < \ldots < s_{k(N)} = \zeta'$ such that the oscillation

$$\omega(f \circ x, [s_i, s_{i+1})) = \sup_{s,s' \in [s_i, s_{i+1})} |f \circ x(s) - f \circ x(s')| \underset{N\to\infty}{\rightarrow} 0,$$

this uniformly in $1 \leq i < k(N)$. Let also $s_i^N = x_N^{-1}(x(s_i))$ be the corresponding times for $f \circ x_N$. We build a time change λ_N (a strictly increasing continuous function) by setting $\lambda_N(s_i) = s_i^N$ for $1 \leq i \leq k(N)$, and interpolating linearly between these times. Easily $|\lambda_N(s_i) - s_i| \leq \sum_{n \notin \pi_1 \cup \ldots \cup \pi_N} (b_n - a_n) \to 0$, and it follows that λ_N converges pointwise and uniformly to the identity function of $[0, \zeta']$. On the other hand, $f \circ x(s_i) = f \circ x_N \circ \lambda_N(s_i)$, so for $s \in (s_i, s_{i+1})$,

$$|f \circ x_N \circ \lambda_N(s) - f \circ x(s)| \leq \omega(f \circ x, [s_i, s_{i+1})) + |f \circ x_N \circ \lambda_N(s) - f \circ x_N \circ \lambda_N(s_i)|.$$

To bound the second term, notice that $x_N((s_i^N, s_{i+1}^N)) \subset x((s_i, s_{i+1})) \cup \bigcup_{n\notin\pi_1\cup\ldots\cup\pi_N}[a_n, b_n]$. Therefore

$$\begin{aligned} |f\circ x_N\circ\lambda_N(s) - f\circ x_N\circ\lambda_N(s_i)| &\leq \omega(f\circ x, [s_i, s_{i+1})) \\ &+ \sup_{n\notin\pi_1\cup\ldots\cup\pi_N}(f(a_n) - f(a_n-) + \omega(f, [a_n, b_n])). \end{aligned}$$

We can conclude that $f\circ x_N\circ\lambda_N$ converges uniformly to $f\circ x$ since the oscillation $\omega(f, [a_n, b_n])$ converges to 0 uniformly in $n\notin\pi_1\cup\ldots\cup\pi_N$ as $N\to\infty$, as does the jump $f(a_n) - f(a_n-)$. $\square$

Under the law $\mathbf{P}^{(-z,\infty)}$ under which X is killed when it first attains $-z$, for every $t>0$ we let $z_1^t \geq z_2^t \geq \ldots \geq 0$ be the marked jumps of X at level t, ranked in decreasing order of magnitude, and τ_i^t be the time of occurrence of the jump with magnitude z_i^t, while σ_i^t is the first time after τ_i^t when X hits level $X_{\tau_i^t-}$ (notice that τ_i^t, σ_i^t are not stopping times). Similarly as before, we let $I_i^t = [\tau_i^t, \sigma_i^t]$.

Lemma 7 *For every $z, t\geq 0$, the process* $\mathtt{UNPLUG}(X, (I_i^t : i\geq 1))$ *has same law as $X^{(t)}$ under $\mathbf{P}$, killed when it first hits $-z$.*

Part of this lemma is that it makes sense to apply the unplugging operation with the intervals I_i^t, that is, that these intervals admit a separated covering maximal sub-family.

Proof. The fact that the intervals I_i^t admit a covering maximal sub-family is obtained by re-using the proof of Lemma 4 and the argument given just after the definition of $C_{i,j}^t$ in the preceding section. Next, write $X = X^{(t)} + Z^{(t)}$. For $a>0$, let $\tau_1^{t,a}$ be the time of the first jump of $Z^{(t)}$ that is $>a$, and let $\sigma_1^{t,a} = \inf\{u\geq\tau_1^{t,a} : X_u = X_{\tau_1^{t,a}-}\}$. Recursively, let $\tau_{i+1}^{t,a} = \inf\{u\geq\tau_i^{t,a} : \Delta Z_u^{(t)} > a\}$ and $\sigma_{i+1}^{t,a} = \inf\{u\geq\tau_{i+1}^{t,a} : X_u = X_{\tau_{i+1}^{t,a}-}\}$. Let $Z_s^{(t,a)} = \sum_{u\leq s}\Delta Z_u^{(t)}\mathbb{1}_{\{\Delta Z_u^{(t)}\leq a\}}$. The $\tau_i^{t,a}$'s are stopping times for the filtration generated by $(X^{(t)}, Z^{(t)})$, as well as the $\sigma_i^{t,a}$'s. By a repeated use of the Markov property at these times we get

$$\mathtt{UNPLUG}(X; (I_i^t : z_i^t > a)) \stackrel{d}{=} X^{(t)} + Z^{(t,a)},$$

where this last process is killed at the time $T_z^{(t,a)}$ when it first hits $-z$. In particular, $T_z - \sum_i(\sigma_i^{t,a} - \tau_i^{t,a})$ has the same law as $T_z^{(t,a)}$, which converges in law to $T_z^{(t)}$ as $a\downarrow 0$ because $Z^{(t,a)}$ converges to 0 uniformly on compact sets, and $X^{(t)}$ enters $(-\infty, -z)$ immediately after $T_z^{(t)}$ by the Markov property and the fact that 0 is a regular point for Lévy processes with infinite total variation. Therefore, writing $|I_i^t|$ for the Lebesgue measure of I_i^t, $T_z - \sum_k' |I_k^t|$ (where the sum is over the I_k^t that are maximal) has same law as $T_z^{(t)}$, and in particular it is nonzero a.s. Now to check that the intervals I_i^t are separated (we are only interested by those which are maximal), consider two left-ends of such intervals such as $\tau_i^{t,a} < \tau_j^{t,a}$ (where a is small enough). The regularity of 0 for the Lévy process X implies that $\inf_{s\in[\sigma_i^{t,a}, \tau_j^{t,a}]} X_s < X_{\sigma_i^{t,a}}$, so by the same arguments as above and the Markov property at $\sigma_i^{t,a}$, there exists a (random) $\varepsilon_{i,j}^a > 0$ such that given $\varepsilon_{i,j}^a$,

$$\tau_j^{t,a} - \sigma_i^{t,a} - \sum_{I_k^t\subset[\sigma_i^{t,a}, \tau_j^{t,a}]} |I_k^t|\mathbb{1}_{\{I_k^t \text{ maximal}\}}$$

is stochastically larger than $T^{(t)}_{\varepsilon^a_{i,j}}$. This ensures the a.s. separation of the I^t_k's, so the a.s. convergence of $\mathtt{UNPLUG}(X,(I^t_i : z^t_i > a))$ to $\mathtt{UNPLUG}(X,(I^t_i, i \geq 1))$ as $a \downarrow 0$ comes from Lemma 6. Identifying the limiting law follows from the above discussion. $\square$

Now let as before $X^t_i(s) = X_{\tau^t_i + s} - X_{\tau^t_i}$ for $0 \leq s \leq \sigma^t_i - \tau^t_i$ and $i \geq 0$, where by convention $\tau^t_0 = 0$, and $\sigma^t_0 = T_1$. We write $-\tau^t_i + I^t_k = [\tau^t_k - \tau^t_i, \sigma^t_k - \tau^t_i]$. The next lemma does most of the job to extract the different tree components of the logged stable tree at time t.

Lemma 8 (i) *Under the law $\mathbf{P}^{(-1,\infty)}$, as $a \downarrow 0$, the processes $\mathtt{UNPLUG}(X^t_i, (-\tau^t_i + I^t_k, k : I^t_k \subsetneq I^t_i \text{ and } z^t_k > a)), i \geq 1$ converge in $\mathbb{D}^\dagger([0,\infty),\mathbb{R})$ to the processes $Y^t_i = \mathtt{UNPLUG}(X^t_i, (-\tau^t_i + I^t_k, k : I^t_k \subsetneq I^t_i)), i \geq 1$.*

(ii) *The process Y^t_i has the same law as $z^t_i + X^{(t)}$ under $\mathbf{P}$, killed when it first hits 0, and these processes are independent conditionally on $(z^t_i, i \geq 1)$.*

(iii) *The sum of the durations of $Y^t_i, i \geq 0$ equals T_1 a.s.*

Proof. (i) Fix $a > 0$, we modify slightly the notations of the preceding proof by letting $\tau^{t,a}_1 < \ldots < \tau^{t,a}_{k(a)}$ be the times when $Z^{(t)}$ accomplishes a jumps that is $> a$, and letting $\sigma^{t,a}_i = \inf\{u \geq \tau^{t,a}_i : X_u = X_{\tau^{t,a}_i -}\}$. Let also $\tau^{t,a}_0 = 0, \sigma^{t,a}_0 = T_1$. Write $I^{t,a}_i = [\tau^{t,a}_i, \sigma^{t,a}_i]$, and let $X^{t,a}_i(s) = X_{\tau^{t,a}_i + s} - X_{\tau^{t,a}_i}$ for $0 \leq s \leq \sigma^{t,a}_i - \tau^{t,a}_i$. By the Markov property at times $\tau^{t,a}_i, \sigma^{t,a}_i$, we obtain that for every i, $X^{t,a}_i$ is independent of $\mathtt{UNPLUG}(X, I^{t,a}_i)$ given the jump $\Delta X_{\tau^{t,a}_i}$. By a repeated use of the Markov property, we obtain the independence of the processes $\mathtt{UNPLUG}(X^{t,a}_i, (-\tau^t_i + I^{t,a}_k : I^{t,a}_k \subsetneq I^{t,a}_i))$ given $(\Delta X_{\tau^{t,a}_i}, 1 \leq i \leq k(a))$, and moreover, the law of $X^{t,a}_i$ given $\Delta X_{\tau^{t,a}_i}$ is that of X under P, killed when it first hits $-\Delta X_{\tau^{t,a}_i}$. Letting $a \downarrow 0$ and applying Lemma 7 finally gives the convergence to the processes Y^t_i, as well as the conditional independence and the distribution of the processes, giving also (ii).

(iii) Let us introduce some extra notation. Say that the marked jump with magnitude z^t_i is of the j-th kind if and only if the future infimum process $(\inf_{s \leq u \leq \tau^t_i} X_u, 0 \leq s \leq \tau^t_i)$ accomplishes exactly j jumps at times that correspond to marked jumps of X. Write $|I^t_i|$ for the duration of X^t_i and let A_j be the set of indices i such that τ^t_i is a jump time of the j-th kind. By a variation of Lemma 4 already used above, every marked jump is of the j-th kind for some j a.s. By Lemma 7 the duration of Y^t_0 is $T_1 - \sum_{i \in A_1} |I^t_i|$, similarly, one has that if $i \in A_j$, the duration of Y^t_i equals $|I^t_i| - \sum_{k \in A_{j+1}} |I^t_k| \mathbb{1}_{\{I^t_k \subset I^t_i\}}$. Therefore, proving that the sum of durations of Y^t_i equals T_1 amounts to showing that $\sum_{i \in A_j} |I^t_i| \to 0$ in probability as $j \to \infty$. But the sum of the marked jumps is finite a.s., since conditionally on a marked jump z^t_i, the duration of the corresponding X^t_i has same law as $T_{z^t_i}$, and since we have independence as i varies. Hence this sum is (conditionally on $(z^t_i, i \geq 1)$) equal in law to $T_{\sum_{i \in A_j} z^t_i}$ under $\mathbf{P}$, and it converges to 0. $\square$

Lemma 9 *The process $(F^+(t), t \geq 0)$ is a Markovian self-similar fragmentation with index $1/\alpha$. Its erosion coefficient is 0*

Proof. For every $v > 0$, define the processes X^t_i under $\mathbf{N}^{(v)}$ as in the preceding section, replacing the duration 1 by v. By virtue of Lemma 8 and by excursion theory, we obtain that for almost every $v > 0$, and for all t in a dense countable subset of $\mathbb{R}_+$, under $\mathbf{N}^{(v)}$, the processes $\mathtt{UNPLUG}(X^t_i, (-\tau^t_i + I^t_k : I^t_k \subsetneq I^t_i \text{ and } z^t_k > a))$ converge as $a \downarrow 0$ to processes Y^t_i that are independent conditionally on the z^t_i's and on their durations, and whose durations

sum to v (by convention we let $X^t_0 = X$). By scaling, this statement remains valid for $v = 1$. We then extend it to all $t \geq 0$ by a continuity argument. The case $t = 0$ is obvious, so take $t_0 > 0$ and $t \uparrow t_0$ in the dense subset of $\mathbb{R}_+$. Almost surely, t_0 is not a time at which a new hub is marked, so $X^{t_0}_i = X^t_i$ for t close enough of t_0, and by Lemma 6 and the fact that $\{I^t_i, i \geq 0\} \subset \{I^{t_0}_i, i \geq 0\}$ for $t \leq t_0$,

$$Y^{t_0}_i = \texttt{UNPLUG}(X^t_i, (-\tau^t_i + I^{t_0}_k : I^{t_0}_k \subsetneq I^{t_0}_i)) = \lim_{t \uparrow t_0} \texttt{UNPLUG}(X^t_i, (-\tau^t_i + I^t_k : I^t_k \subsetneq I^t_i)).$$

Now recall the notation $X^t_{i,j}, I^t_{i,j} = [\tau^t_{i,j}, \sigma^t_{i,j}]$ from Sect. 3.3, and for $j \geq 1$ write $Y^t_{i,j} = \texttt{UNPLUG}(X^t_{i,j}, (-\tau^t_{i,j} + I^t_k : I^t_k \subsetneq I^t_{i,j}))$ for the excursions of Y^t_i above its infimum, ranked in the order corresponding to $X_{i,j}$. Then by the same arguments as in the proof of Lemma 7, the joint law of the durations of $Y^t_0, Y^t_{i,j}, i \geq 1, j \geq 1$ equals the law of $(|C^t_0|, |C^t_{i,j}|, i \geq 1, j \geq 1)$ with notations above. Hence, by Lemma 5 and the fact that excursions of $X^{(t)}$ with prescribed duration are stable excursions, it holds that conditionally on $F^+(t) = (x_1, x_2, \ldots)$, the excursions $Y^t_{i,j}$ are independent stable excursions with respective durations $x_1, x_2, \ldots$.

Now let $\sim^{t,i,j}_{t'}$ be the equivalence relation defined for the excursion $Y^t_{i,j}$ in a similar way as $\sim_t$ for the normalized excursion of X. Write also $j_t(u) = u - \tau^t_{i,j} - \sum_{k: I^t_k \subsetneq I^t_i, \sigma^t_k < u} |I^t_k|$ for $u \in [0, 1]$, whenever $u \in C^t_{i,j}$. Then it is clear that if $x, y \in C^t_{i,j}$, one has also $x \sim_{t+t'} y$ if and only if $j_t(x) \sim^{t,i,j}_{t'} j_t(y)$. By the scaling property, a stable excursion ε^x with duration x where every jump with magnitude ℓ is marked with probability $1 - \exp(-t'\ell)$ is obtained by taking a normalized excursion $(\varepsilon^1_s, 0 \leq s \leq 1)$, marking every jump with magnitude ℓ independently with probability $1 - \exp(-t' x^{1/\alpha} \ell)$, and then letting $\varepsilon^x_s = x^{1/\alpha} \varepsilon^1_{s/x}$ for $0 \leq s \leq x$; the marked jumps of ε^x occurring at the times sx whenever s is a marked jump time for ε^1. This means that given $F^+(t) = (x_1, \ldots)$, the process $(F^+(t + t'), t' \geq 0)$ has the same law as $((x_1 F^{+,1}(x_1^{1/\alpha} t'), x_2 F^{+,2}(x_2^{1/\alpha} t'), \ldots)^\downarrow, t' \geq 0)$ where the $F^{+,i}$'s are independent copies of F^+. This entails both the Markov property and the self-similar property, the self-similarity index being $1/\alpha$. Moreover, Lemma 8 (iii) shows that the sum of durations of $Y^t_{i,j}$ is 1 a.s. under $\mathbf{N}^{(1)}$, so $\sum_i F^+_i(t) = 1$ a.s. and the erosion coefficient must be 0 according to [9].

To conclude, we notice that the previous result of continuity in probability of F^+ at time 0 extends to any time $t \geq 0$ by the self-similar fragmentation property. $\square$

4.2 Splitting rates and dislocation measure

To complete the study of the characteristics of F^+, we must identify the dislocation measure. This is done by computing the *splitting rate* of the stable tree, that is, the rate at which the tree with mass 1 instantaneously splits into a sequence of subtrees with given masses $s_1 \geq s_2, \ldots$ with $\sum_i s_i = 1$, by analogy with the splitting rate of the Brownian CRT in [3].

We will need the following lemma from [22], which is similar to Lévy's method to compute the jump measure of a Lévy process.

Lemma 10 *Let $(F(t), t \geq 0)$ be a self-similar fragmentation with index $\beta \geq 0$ and erosion coefficient $c = 0$. Then for every function G that is continuous and null on a neighborhood of $(1, 0, \ldots)$ in S,*

$$t^{-1} E[G(F(t))] \underset{t \downarrow 0}{\to} \nu(G).$$

Recall that our marking process on the hubs of the tree amounts to taking a Poisson process with intensity $m(\mathrm{d}v) = \sum_b L(b)\delta_b(\mathrm{d}v)$ on $\mathcal{T}$, where the sum is over hubs $b \in \mathcal{T}$. For $v \in \mathcal{T}$, let $\mathcal{T}_1(v), \mathcal{T}_2(v), \ldots$ be the tree components of the forest obtained when removing v, arranged by decreasing order of masses, and let

$$r(\mathrm{d}\mathbf{s}) = N^{(1)}\left(m\{v \in \mathcal{T} : (\mu(\mathcal{T}_1(v)), \mu(\mathcal{T}_2(v)), \ldots) \in \mathrm{d}\mathbf{s}\}\right)$$

be the rate at which a m-picked vertex splits $\mathcal{T}$ into trees with masses in a volume element $\mathrm{d}\mathbf{s}$ (recall that the stable tree is defined under the normalized excursion law $N^{(1)}$). It is quite intuitive that the splitting rate equals the dislocation measure of F^+, and Theorem 1 reduces to the two following lemmas :

Lemma 11 *The splitting rate $r(\mathrm{d}\mathbf{s})$ equals the dislocation measure ν_+ of F^+.*

Proof. For $t \geq 0$ we let $\mathcal{T}(t)$ be the forest obtained by our logging procedure of the stable tree at time t. Let $n \geq 2$, and consider n leaves $L_1, \ldots, L_n \in \mathcal{T}$ that are independent and distributed according to the mass measure μ, conditionally on μ (we are implicitly working on an enlarged probability space). Write $\Pi_n(t)$ for the partition of $[n] = \{1, \ldots, n\}$ obtained by letting i and j be in the same block of $\Pi_n(t)$ if and only if L_i and L_j belong to the same tree component of $\mathcal{T}(t)$. For $K > 2$ let $\Lambda^n_K(t)$ be the event that at time t, the leaves $L_1, \ldots, L_n$ are all contained in tree components of $\mathcal{T}(t)$ with masses $> 1/K$. Write $\mathcal{P}^*_n$ for the set of partitions π of $[n] = \{1, \ldots, n\}$ with at least two non void blocks $A_1, \ldots, A_k$ (for some arbitrary ordering convention). Given $F^+(t) = \mathbf{s} = (s_1, s_2, \ldots)$, the probability that $\Pi_n(t)$ equals some partition $\pi \in \mathcal{P}^*_n$ and that $\Lambda^n_K(t)$ happens is

$$G_K(\mathbf{s}) = \mathbf{N}^{(1)}(\Pi_n(t) = \pi, \Lambda^n_K(t) | F^+(t) = \mathbf{s}) = \sum_{i_1, \ldots, i_k}^{*K} \prod_{j=1}^{k} s_{i_j}^{\# A_j},$$

the sum being over pairwise distinct i_j's such that $s_{i_j} > 1/K$. This last function is continuous and null on a neighborhood of $(1, 0, \ldots)$, so Lemma 10 (which we may use by Lemma 9) gives

$$\lim_{t \downarrow 0} t^{-1} \mathbf{N}^{(1)}(\Pi_n(t) = \pi, \Lambda^n_K(t)) = \int_S \nu_+(\mathrm{d}\mathbf{s}) \sum_{i_1, \ldots, i_k}^{*K} \prod_{j=1}^{k} s_{i_j}^{\# A_j}. \tag{8}$$

We claim that knowing this quantity for every n, π, K characterizes ν_+. One can obtain this by first letting $K \to \infty$ by monotone convergence, and then using an argument based on exchangeable partitions as in [18, p. 378] (a Stone-Weierstrass argument can also work).

On the other hand, for any b in the set $\mathcal{H}(\mathcal{T})$ of branchpoints of $\mathcal{T}$, let π^b_n be the partition of $[n]$ obtained by letting i and j be in the same block if and only if b is not on the path from L_i to L_j. Let also $\mathcal{T}_{L_i}(b)$ be the tree component of the forest obtained by removing b from $\mathcal{T}$ that contains L_i. For $K \in (2, \infty]$ and $\pi \in \mathcal{P}^*_n$, let $\Psi^n_K(\pi)$ be the set of branchpoints $b \in \mathcal{T}$ such that $\pi^b_n = \pi$ and such that $\mu(\mathcal{T}_{L_i}(b)) > 1/K$ for $1 \leq i \leq n$, and let $\Psi^n_K = \bigcup_{\pi \in \mathcal{P}^*_n} \Psi^n_K(\pi)$. Recall that we may construct the fragmentation F^+ by cutting the stable tree at the points of a Poisson point process $(b(s), s \geq 0)$ with intensity $\mathrm{d}s \otimes m(\mathrm{d}b)$. Now for $\Pi_n(t) = \pi$ to happen, it is plainly necessary that at least one $b(s)$ falls in Ψ^n_∞ for some $s \in [0, t]$, if in addition $\Lambda^n_K(t)$ happens then no $b(s), 0 \leq s \leq t$ must fall in $\Psi^n_\infty \setminus \Psi^n_K$.

Therefore,

$$\mathbf{N}^{(1)}(\Pi_n(t) = \pi, \Lambda_K^n(t)) = \mathbf{N}^{(1)}\left(\exists!\, s \in [0,t] : b(s) \in \Psi_\infty^n, \text{ and } b(s) \in \Psi_K^n(\pi), \Lambda_K^n(t)\right) + R(t), \tag{9}$$

where the residual $R(t)$ is bounded by the probability that $b(s)$ falls in Ψ_∞^n for at least two $s \in [0,t]$. Hence $R(t) = o(t)$ by standard properties of Poisson processes provided we can show that $N^{(1)}[m(\Psi_\infty^n)] < \infty$. This could be shown using the forthcoming lemma, but we may also just notice that if $N^{(1)}[m(\Psi_\infty^n)]$ was infinite, then there would be arbitrarily many $b(s), 0 \le s \le t$ falling in $\Psi_\infty^n \setminus \Psi_K^n$ for some appropriately large K, and the probability in (9) would be 0, which is impossible from the beginning of this proof and since F^+ is a self-similar fragmentation with nonzero dislocation measure (because it has erosion coefficient 0 and it is not constant). On the other hand, conditionally on the event on the right-hand side of (9), the $b(s), 0 \le s \le t$ that do not fall in Ψ_∞^n (call them $b'(s)$) form an independent Poisson point process with intensity $m(\cdot \cap \mathcal{H}(\mathcal{T}) \setminus \Psi_\infty^n)$. Therefore, the size of the tree component of the forest obtained when removing the points $b'(s), 0 \le s \le t$ that contains L_1 converges a.s. to 1 as $t \downarrow 0$ (so it also contains the other L_i's for small t a.s.), as it is stochastically bigger than the component of $\mathcal{T}(t)$ containing L_1, and since $F^+(t) \to (1, 0, \ldots)$ in probability as $t \downarrow 0$. It follows that one can remove $\Lambda_K^n(t)$ from the right-hand side of (9), and basic properties of Poisson measures finally give $t^{-1}\mathbf{N}^{(1)}(\Pi_n(t) = \pi, \Lambda_K^n(t)) \to \mathbf{N}^{(1)}[m(\Psi_K^n(\pi))] = N^{(1)}[m(\Psi_K^n(\pi))]$. This last quantity is finally equal to $\int_S r(\mathrm{d}\mathbf{s}) \sum_{i_1,\ldots,i_k}^{*K} \prod_{j=1}^k s_{i_j}^{\#A_j}$ since L_i belongs to $B \subset \mathcal{T}$ with probability $\mu(B)$ that is equal to the Lebesgue measure of the subset of $[0,1]$ encoding B. Identifying with (8) gives the claim. $\square$

Lemma 12 *One has $r(\mathrm{d}\mathbf{s}) = \nu_\alpha(\mathrm{d}\mathbf{s})$ with the notations of Theorem 1.*

Proof. We must see what is the effect of splitting $\mathcal{T}$ at a hub b picked according to $m(\mathrm{d}v)$. By definition, m picks a hub proportionally to its local time, and by Proposition 2, hubs are in one-to-one correspondence with jumps of the stable excursion with duration 1. More precisely, if b is the hub that has been picked and with the notations $\tau(b), \sigma(b)$ above, the masses of the tree components obtained when removing b are equal to the lengths of the constancy intervals of the infimum process of $(X_{\tau(b)+s} - X_{\tau(b)}, 0 \le s \le \sigma(b) - \tau(b))$, and the extra term $1-(\sigma(b)-\tau(b))$. By Vervaat's theorem, we may suppose that the excursion is the Vervaat transform of a stable bridge and that the marked jump in the excursion corresponds to a jump $(s, \Delta X_s)$ of the bridge picked according to the σ-finite measure $\sum_{u:\Delta X_u>0} \Delta X_u \delta_{(u,\Delta X_u)}(\mathrm{d}s, \mathrm{d}x)$. By Lemma 2, this marked jump equals (s,x) according to a certain σ-finite "law", while given $(s, \Delta X_s) = (s,x)$, the bridge X has the same law as $X \oplus (s,x)$, under the law $P^1_{0\to -x}$.

Therefore, we have have obtained a representation of the excursion together with a marked jump as a bridge X with law $P^1_{0\to -x}$, where x is independent with some σ-finite "law", to which has been added the marked jump of size x at an independent uniform time s, and which has finally undergone the Vervaat transformation. Using the invariance of bridge laws under independent cyclic shifts, it is now easy to see that the lengths of the constancy intervals of $(X_{\tau(b)+s} - X_{\tau(b)}, 0 \le s \le \sigma(b) - \tau(b))$ defined above have the same law as the intervals of constancy of the infimum process of $(X_{s'+s} - X_s, 0 \le s' \le T_x)$ under $P^1_{0\to -x}$ (with x as above), while the remaining term $1 - (\sigma(b) - \tau(b))$ has (jointly) law $1 - T_x$.

It is now easy that conditionally on $x, T_x = t$ these constancy intervals have the same law as $\Delta T_{[0,x]}$ given $T_x = t$ under P (one actually checks that $(X_u, 0 \leq u \leq T_x)$ is the first-passage bridge with law $P^t_{0\downarrow -x}$ defined before Lemma 15 below). The law of $1 - T_x$ given x is simply obtained by using the definition of bridges and the Markov property : for $a < 1$ and positive measurable f,

$$\begin{aligned} E^1_{0\to -x}[f(1-T_x)\mathbb{1}_{\{T_x<a\}}] &= E^1[f(1-T_x)\mathbb{1}_{\{T_x<a\}}p_1(-x)^{-1}p_{1-a}(-x-X_a)] \\ &= \int_0^a \mathrm{d}s\, q_x(s) f(1-s) p_1(-x)^{-1} \int \mathrm{d}y\, p_{a-s}(y) p_{1-a}(-y) \\ &\underset{a\to 1}{\to} \int_0^1 \mathrm{d}s\, q_x(s) f(1-s) p_{1-s}(0) p_1(-x)^{-1}. \end{aligned}$$

In the last integral, change variables $1 - s \to s$, use $p_1(-x) = x^{-1}q_x(1)$, check by scaling that $p_s(0) = s^{-1/\alpha}p_1(0)$, and conclude by identifying with Lemma 1 that $1 - T_x$ under $P^1_{0\to -x}$ has same law as a size-biased pick from $\Delta T_{[0,x]}$ given $T_x = 1$ under P (notice that in particular we must have $p_1(0) = c_\alpha$). By Lemma 1 (ii), it follows that given the local time x of the marked hub b, the law of the sizes of the stable tree split at this hub is the same as that of $\Delta T_{[0,x]}$ given $T_x = 1$ under P.

Putting pieces together and recalling the distribution of the marked jump x from Lemma 2 we obtain the formula

$$r(\mathrm{d}\mathbf{s}) = \int_0^\infty \mathrm{d}x \frac{C_\alpha p_1(-x)}{x^\alpha p_1(0)} P(\Delta T_{[0,x]} \in \mathrm{d}\mathbf{s} | T_x = 1).$$

By using the scaling property for T and its density ($q_x(1) = x^{-\alpha}q_1(x^{-\alpha})$), formula (3) and a change of variables, we obtain

$$\begin{aligned} r(\mathrm{d}\mathbf{s}) &= \int_0^\infty \mathrm{d}x \frac{C_\alpha q_1(x^{-\alpha})}{c_\alpha x^{2\alpha+1}} P(T_1^{-1}\Delta T_{[0,1]} \in \mathrm{d}\mathbf{s} | T_1 = x^{-\alpha}) \\ &= \alpha^{-1}c_\alpha^{-1}C_\alpha \int_0^\infty \mathrm{d}u\, u\, q_1(u) P(T_1^{-1}\Delta T_{[0,1]} \in \mathrm{d}\mathbf{s} | T_1 = u), \end{aligned}$$

which gives the desired formula, after checking that $\alpha^{-1}c_\alpha^{-1}C_\alpha = D_\alpha$. $\square$

5 Study of $F^\natural$

Recall the construction of $F^\natural$ (under the measure $\mathbf{N}^{(1)}$) from Sect. 1. As noticed above, this fragmentation process somehow generalizes the one considered in [7, 20] (we could actually build it in an analogous way for a large class of Lévy processes with no negative jumps, though the resulting fragmentations would not be self-similar due to the absence of scaling). Notice that none of the fragmentation processes of [20] are self-similar, but for the Brownian case. The reason for this was a lack of a Girsanov-type theorem saying that a Lévy process plus drift has a law that is absolutely continuous with the initial process, but for the Brownian case. Here, this is fixed by Proposition 1, but where the operation is removing jumps rather than adding a drift.

5.1 The self-similar fragmentation property

For any $t' > t \geq 0$ let $\mu_t(x, \mathrm{d}\mathbf{s})$ be a kernel from $\mathbb{R}_+^*$ to S defined as follows : $\mu_t(x, \mathrm{d}\mathbf{s})$ is the law of the ranked lengths of the constancy intervals of the process $\underline{X}^{(t)}$ under $\mathbf{N}^{(x)}$. Moreover, define $F^{\natural,1}$ exactly as $F^\natural$, but where X is under the law $\mathbf{P}^{(-1,\infty)}$. In particular, $F^{\natural,1}(t)$ is not S-valued (the sum of its components is random).

Proposition 3 (i) *The processes $F^{\natural,1}$ and $F^\natural$ enjoy the fragmentation property, with fragmentation kernel $\mu_t(x, \mathrm{d}\mathbf{s})$. That is, conditionally on $F^{\natural,1}(t) = (x_1, x_2, \ldots)$ (resp. $F^\natural(t)$), $F^{\natural,1}(t+t')$ (resp. $F^\natural(t+t')$) has the same law as the decreasing rearrangement of independent sequences $\mathbf{s}_i$ with respective laws $\mu_{t'}(x_i, \mathrm{d}\mathbf{s})$.*

(ii) *The process $F^\natural$ is a self-similar fragmentation with index $1/\alpha$, and no erosion.*

The fact that $F^\natural$ is a fragmentation process directly comes from the fact that the processes $X^{(t')} - X^{(t)} = Z^{(t)} - Z^{(t')}$ are non-increasing. We now prove the fragmentation property. The key lies in a Skorokhod-like relation that is analogous to that in [7] and generalized in [20].

Lemma 13 *For every $t, t' \geq 0$ and $s \geq 0$, one has*

$$\underline{X}_s^{(t)} = \inf_{0 \leq u \leq s} (\underline{X}_u^{(t+t')} + (Z_u^{(t+t')} - Z_u^{(t)})).$$

The proof can be done following exactly the same lines as in [7, Lemma 2]. As a consequence, we obtain that the sigma-field $\mathcal{G}_t = \sigma\{X^{(t)}, (Z^{(s)}, 0 \leq s \leq t)\}$ induces a filtration, with respect to which $F^{\natural,1}$ is adapted.

The end of the proof of the fragmentation property in Proposition 3 also goes as in [7]. For any variable K that is $\mathcal{G}_t$-measurable, the excursions of $X^{(t)}$ above its infimum and before time $T_K^{(t)}$ are independent excursions conditionally on $\mathcal{G}_t$, respectively conditioned to have durations $\ell_{1,X}^{(t)}, \ell_{2,X}^{(t)}, \ldots$ where the last family is the decreasing sequence of constancy intervals of $\underline{X}^{(t)}$ before time $T_K^{(t)}$. Take $K = \underline{X}_{T_1}^{(t)}$, which is measurable with respect to $\mathcal{G}_t$ by virtue of the Skorokhod property. Then $T_K^{(t)} = T_1$, which gives readily that conditionally on $\mathcal{G}_t$, the excursions of $X^{(t)}$ above $\underline{X}^{(t)}$ are independent with durations $(F_i^{\natural,1}(t), i \geq 0)$.

To conclude, it remains to notice that the lack of memory of the exponential law implies that the jumps that are unmarked at time t but that are marked at time $t + t'$ can be obtained also by marking with probability $1 - e^{-t'\ell}$ any unmarked jump at time t that has magnitude ℓ. Thus, conditionally on $F^{\natural,1}(t)$, we obtain a sequence with the same law as $F^{\natural,1}(t+t')$ by taking independent sequences $(\mathbf{s}_i, i \geq 1)$ with laws $\mu_{t'}(F_i^{\natural,1}(t), \mathrm{d}\mathbf{s})$ and rearranging, as claimed. This remains true for $F^\natural$ by excursion theory and scaling.

To show the self-similarity for $F^\natural$, it then suffices to check, using the scaling property of the excursions of stable processes, that $\mu_t(x, \mathrm{d}\mathbf{s})$ is the image of $\mu_{tx^{1/\alpha}}(1, \mathrm{d}\mathbf{s})$ by $\mathbf{s} \mapsto x\mathbf{s}$. The fact that $F^\natural$ has no erosion again comes from the fact that $\sum_i F_i^\natural(t) = 1$ a.s.

5.2 The semigroup

According to the preceding section, and since plainly there is no loss of mass in the fragmentation $F^\natural$ (so the erosion coefficient is 0), proving Theorem 2 requires only to check that the dislocation measure of $F^\natural$ equals that of F^+. It is intuitively straightforward that this is the case, by looking at the procedure we use for deleting jumps, and indeed we

could easily follow the same lines as above and compute a "splitting rate" for the bridge, when the "first" marked jump is deleted. However, a nice feature of this fragmentation is that we can compute explicitly its semigroup (hence that of F^+), as will follow. The semigroup then gives enough information to re-obtain the dislocation measure, and this will prove Theorem 2. Recall from Sect. 2 that $\rho_1^{(t)}$ is the density of $Z_1^{(t)}$ under $\mathbf{P}$.

Proposition 4 *The semigroup of $F^\natural$ is given by*

$$\mathbf{N}^{(1)}(F^\natural(t) \in \mathrm{d}\mathbf{s}) = \int_0^\infty \mathrm{d}z \frac{p_1^{(t)}(-z)\rho_1^{(t)}(z)}{p_1(0)} P(\Delta T_{[0,z]} \in \mathrm{d}\mathbf{s} | T_z = 1).$$

We will need a couple of intermediate lemmas. Since $Z^{(t)}$ is non-decreasing, under the law $\mathbf{N}^{(1)}$, the process $X^{(t)}$ starts at 0 and hits $-Z_1^{(t)}$ at time 1 for the first time. Since we are interested in the constancy intervals of $\underline{X}^{(t)}$, and thanks to Vervaat's theorem, we would like to relate these constancy intervals to the bridge of X. We now work under the law of the bridge with unit duration $\mathbf{P}^1_{0\to 0}$, so we may suppose that the excursion of X with duration 1 is equal to the Vervaat transform VX. Let $m = -\underline{X}_1$ be the absolute value of the minimum of X, and $\tau_2 = T_{m-}$ be the (a.s. unique) time when X attains this minimum, so $VX = V^{\tau_2}X$. Decompose X as $X^{(t)} + Z^{(t)}$ where $Z^{(t)}$ is the cumulative process of marked jumps. Then $VX = V^{\tau_2}X^{(t)} + V^{\tau_2}Z^{(t)}$, and by independence of the marking procedure of jumps we can consider that $V^{\tau_2}Z^{(t)}$ is the cumulative process of marked jumps for the excursion VX. The problem is now to describe the law of lengths of the constancy intervals of the process $\underline{V^{\tau_2}X^{(t)}}$. Let $m^{(t)} = -\underline{X}_1^{(t)}$ be the absolute value of the minimum of $X^{(t)}$ and $\tau_3 = T^{(t)}_{m^{(t)}-}$ be the (a.s. unique) time when $X^{(t)}$ attains this minimum. Let also $\tau_1 = T^{(t)}_{m^{(t)}-Z_1^{(t)}}$ be the first time when $X^{(t)}$ attains the value $Z_1^{(t)} - m^{(t)}$. The following lemma is somehow "deterministic". For $a < b$, write $X_{[a,b]}$ for the process $(X_{a+s} - X_a, 0 \leq s \leq b-a)$.

Lemma 14 *One has $\tau_1 \leq \tau_2 \leq \tau_3$ a.s., and the sequence of lengths of the constancy intervals of $\underline{V^{\tau_2}X}^{(t)}$, ranked in decreasing order, is equal to that of the process $\underline{X}^{(t)}_{[\tau_1,\tau_3]}$, to which has been added (at the appropriate rank) the extra term $1 - \tau_3 + \tau_1$.*

Proof. Since $Z^{(t)}$ is an increasing process, one has $X^{(t)}_{\tau_2} = X_{\tau_2} - Z^{(t)}_{\tau_2} \leq X_s - Z^{(t)}_s$ for any $s \leq \tau_2$. Hence, $X^{(t)}_{\tau_2} = \underline{X}^{(t)}_{\tau_2}$ which implies $\tau_2 \leq \tau_3$. On the other hand, one has $-m^{(t)} = X_{\tau_3} - Z^{(t)}_{\tau_3} \geq -m - Z_1^{(t)}$ and thus $m^{(t)} - Z_1^{(t)} \leq m$, implying $\tau_1 \leq \tau_2$.

For convenience, if $(f(x), 0 \leq x \leq \zeta)$ and $(f'(x), 0 \leq x \leq \zeta')$ are two càdlàg functions, we let $f \bowtie f'$ be the concatenation of the paths of f and f', defined by

$$f \bowtie f'(s) = \begin{cases} f(s) & \text{if } 0 \leq s < \zeta \\ f'(s-\zeta) + f(\zeta) & \text{if } \zeta \leq s \leq \zeta + \zeta' \end{cases}.$$

We let $Y^1 = X^{(t)}_{[0,\tau_2]}$, $Y^2 = X^{(t)}_{[\tau_2,\tau_3]}$ and $Y^3 = X^{(t)}_{[\tau_3,1]}$, so $X^{(t)} = Y^1 \bowtie Y^2 \bowtie Y^3$, and $V^{\tau_2}X^{(t)} = Y^2 \bowtie Y^3 \bowtie Y^1$.

Observing that Y_3 is non-negative, we obtain that $\underline{Y^2 \bowtie Y^3} = \underline{Y}^2 \bowtie \mathbf{0}_{[0,1-\tau_3]}$ where $\mathbf{0}_{[0,a]}$ is the null process on $[0,a]$. Since the final value of Y_3 is $m^{(t)} - Z_1^{(t)}$, we obtain that

$$\underline{V^{\tau_2}X}^{(t)} = \underline{Y}^2 \bowtie \mathbf{0}_{[0,1-\tau_3]} \bowtie \mathbf{0}_{[0,\tau_1]} \bowtie \underline{X}^{(t)}_{[\tau_1,\tau_2]} = \underline{Y}^2 \bowtie \mathbf{0}_{[0,1-\tau_3+\tau_1]} \bowtie \underline{X}^{(t)}_{[\tau_1,\tau_2]}.$$

It follows that the constancy intervals of $\underline{V^{\tau_2}X}^{(t)}$ are the same as those of $\underline{X}^{(t)}$, except for the first and last constancy intervals of $\underline{X}^{(t)}$ which are merged to form the constancy interval with length $1-\tau_3+\tau_1$. $\square$

The rest of the section is devoted to the study of these constancy intervals. Recall from Lemma 3 that under $\mathbf{P}^1_{0\to 0}$, the process $X^{(t)}$ has law $P^1_{0\to -\mathcal{Z}}$, where $\mathcal{Z}$ is an independent random variable with law $P(\mathcal{Z}\in \mathrm{d}z)=p_1(0)^{-1}p_1^{(t)}(-z)\rho_1^{(t)}(z)\mathrm{d}z$. It thus suffices to analyze the constancy intervals of $\underline{X}_{[\tau_1,\tau_3]}$ under the law $P^1_{0\to -z}$ for fixed $z>0$, where we now call $m=-\underline{X}_1$, τ_1 the time when X first hits level $z-m$ and τ_3 the first time when X attains level $-m$.

For $z>0$, let $(P^v_{0\downarrow -z}, v>0)$ be a regular version of the conditional law $P^{(-z,\infty)}[\cdot|T_z=v]$. Call this the law of the first-passage bridge from 0 to $-z$ with length v. A consequence of the Markov property is

Lemma 15 *Let $a,b>0$. For (Lebesgue) almost every $v>0$, under the law $P^v_{0\downarrow -(a+b)}$, the law of T_a is given by*

$$P^v_{0\downarrow -(a+b)}(T_a\in \mathrm{d}s)=\mathrm{d}s\frac{q_a(s)q_b(v-s)}{q_{a+b}(v)}.$$

Moreover, conditionally on T_a, the paths $(X_s, 0\le s\le T_a)$ and $(X_{s+T_a}-a, 0\le s\le T_{a+b}-T_a)$ are independent with respective laws $P^{T_a}_{0\downarrow -a}$ and $P^{v-T_a}_{0\downarrow -b}$.

We also state a generalization of Williams' decomposition of the excursion of Brownian motion at the maximum, given in Chaumont [11]. We need to make a step out of the world of probability and consider σ-finite measures instead of probability laws. Recall that $m_v=-\underline{X}_v$ is the absolute value of the minimum before time v, and with our notations T_{m_v-} is the first time (and a.s. last before v) when X attains this value. Write

$$\begin{aligned}\underleftarrow{X}_s &= X_s & 0\le s\le T_{m_v-},\\ \underrightarrow{X}_s &= m_v+X_{s+T_{m_v-}} & 0\le s\le v-T_{m_v-}\end{aligned}$$

for the pre- and post- minimum processes of X before time v. Then by [11],

Lemma 16 *One has the identity for σ-finite measures*

$$\int_0^\infty \mathrm{d}v P^v(\underleftarrow{X}\in \mathrm{d}\omega, \underrightarrow{X}\in \mathrm{d}\omega')=\int_0^\infty \mathrm{d}x P^{(-x,\infty)}(\mathrm{d}\omega)\otimes \int_0^\infty \mathrm{d}u N^{>u}(\mathrm{d}\omega'),$$

where $N^{>u}$ is the finite measure characterized by $N^{>u}(F(X))=N(F(X_s,0\le s\le u),\zeta(X)>u)$ for every non-negative measurable F. This in turn determines entirely the laws P^v for $v>0$.

Loosely speaking, if v is "random" with "law" the Lebesgue measure on $(0,\infty)$, the pre- and post- minimum processes are independent with respective "laws" $\int_0^\infty \mathrm{d}x P^{(-x,\infty)}(\mathrm{d}\omega)$ and $\int_0^\infty \mathrm{d}u N^{>u}(\mathrm{d}\omega)$. As a consequence of this identity, we have that under P^v for some fixed $v>0$, conditionally on m_v and $T_{m_v-}=\tau$, the processes $\underleftarrow{X}$ and $\underrightarrow{X}$ are independent with respective laws $P^\tau_{0\downarrow -m_v}(\mathrm{d}\omega)$ and $(N^{>v-\tau}(1))^{-1}N^{>v-\tau}(\mathrm{d}\omega')$.

Lemma 17 *Let $z>0$. Under the probability $P^1_{0\to -z}$, conditionally on $\tau_3-\tau_1=t$, the ranked sequence of lengths of the constancy intervals of the infimum process of $(X_{s+\tau_1}, 0\le s\le \tau_3-\tau_1)$ have the same law as $\Delta T_{[0,z]}$ given $T_z=t$ under P.*

Proof. We first condition by the value of (m,τ_3). Then by Lemma 16 the path $\underleftarrow{X}$ has the law $P^{\tau_3}_{0\downarrow -m}$ of the first-passage bridge from 0 to $-z$ with lifetime τ_3. Applying Lemma 15 and the Markov property we obtain that conditionally on τ_1 the path $(X_{s+\tau_1}+m-z, 0\leq s\leq \tau_3-\tau_1)$ is a first passage bridge ending at $-z$ at time $\tau_3-\tau_1$. Since it depends only on $\tau_3-\tau_1$, we have obtained the conditional distribution given $\tau_3-\tau_1$. Hence, the sequence defined in the lemma's statement has the same conditional law as the ranked lengths of the constancy intervals of the infimum process of such a first-passage bridge, that is, it has the same law as $\Delta T'_{[0,z]}$ given $T'_z=\tau_3-\tau_1$, with T' as in the statement. $\square$

The last lemma gives an explicit form for the law of the remaining length $1-\tau_3+\tau_1$ under $P^1_{0\to -z}$.

Lemma 18 *One has*

$$P^1_{0\to -z}(1-\tau_3+\tau_1\in \mathrm{d}s)=\mathrm{d}s\frac{c_\alpha z q_z(1-s)}{s^{1/\alpha}q_z(1)},$$

which is the law of a size-biased pick of the sequence $\Delta T_{[0,z]}$ given $T_z=1$ under P.

Proof. By Lemma 16, if s is "distributed" according to Lebesgue measure on $\mathbb{R}_+$, then under P^s, the processes $\underleftarrow{X}$ and $\underrightarrow{X}$ are independent with respective "laws" $\int_0^\infty \mathrm{d}xP^{(-x,\infty)}(\mathrm{d}\omega)$ and $\int_0^\infty \mathrm{d}uN^{>u}(\mathrm{d}\omega')$. Our first task is to disintegrate these laws to obtain a relation under $P^1_{0\to -z}$. Let H and H' be two continuous bounded functionals and f be continuous with a compact support on $(0,\infty)$. Then, letting $T^\omega_\cdot=\inf\{s\geq 0:\omega(s)<\cdot\}$,

$$\begin{aligned}
&\int_0^\infty \mathrm{d}sf(s)E^s[H(\underleftarrow{X})H'(\underrightarrow{X})\,|\,|X_s+z|<\varepsilon]\\
&=\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}u\iint P^{(-x,\infty)}(\mathrm{d}\omega)N^{>u}(\mathrm{d}\omega')f(T^\omega_x+u)H(\omega)H'(\omega')\frac{\mathbb{1}_{\{|z-x+\omega'(u)|<\varepsilon\}}}{P(|X_{T^\omega_x+u}+z|<\varepsilon)}\\
&=\int_0^\infty \mathrm{d}u\int N^{>u}(\mathrm{d}\omega')H'(\omega')\int_0^\infty \mathrm{d}x\frac{\mathbb{1}_{\{|z-x+\omega'(u)|<\varepsilon\}}}{2\varepsilon}\int P^{(-x,\infty)}(\mathrm{d}\omega)\frac{f(T^\omega_x+u)H(\omega)}{(2\varepsilon)^{-1}P(|X_{T^\omega_x+u}+z|<\varepsilon)}.
\end{aligned}$$

The measure $(2\varepsilon)^{-1}\mathbb{1}_{\{|z-x+\omega'(u)|<\varepsilon\}}\mathrm{d}x$ converges weakly as $\varepsilon\to 0$ to the Dirac mass at $z+\omega'_u$. Recall that the family of probability measures $P^{(-x,\infty)}$ is continuous as x varies. Since f has compact support, we can restrain T^ω_x+u to stay in a compact set. Then, the denominator in the last integral, which converges to $p_{T^\omega_x+u}(-z)$, remains bounded and converges uniformly in x and u. Then the boundedness of H implies that the two last integrals converge to

$$\int P^{(-z-\omega'_u,\infty)}(\mathrm{d}\omega)\frac{f(T^\omega_{z+\omega'(u)}+u)}{p_{T^\omega_x+u}(-z)}.$$

Now, the measure $N^{>u}$ is a finite measure, so the fact that u actually stays in a compact set and the fact that the two last integrals above remain bounded allow to apply the dominated convergence theorem to obtain

$$\begin{aligned}
&\int_0^\infty \mathrm{d}sf(s)P^s_{0\to -z}(H(\underleftarrow{X})H'(\underrightarrow{X}))\\
&=\int_0^\infty \mathrm{d}u\int N^{>u}(\mathrm{d}\omega')H'(\omega')\int P^{(-z-\omega'(u),\infty)}(\mathrm{d}\omega)H(\omega)\frac{f(T_{z+\omega'(u)}(\omega)+u)}{p_{T^\omega_x+u}(-z)}
\end{aligned}$$

Now we disintegrate this relation by taking $f(s) = (2\varepsilon)^{-1}\mathbb{1}_{[1-\varepsilon,1+\varepsilon]}(s)$, so a similar argument as above gives that the left hand side converges to $P^1_{0\to -z}(H(\underleftarrow{X})H'(\underrightarrow{X}))$ as $\varepsilon \downarrow 0$, whereas the right hand side is

$$\int_0^\infty \mathrm{d}u \int N^{>u}(\mathrm{d}\omega')H'(\omega') \int P^{(-z-\omega'(u),\infty)}(\mathrm{d}\omega)H(\omega)\frac{\mathbb{1}_{[1-\varepsilon,1+\varepsilon]}(T^\omega_{z+\omega'(u)}+u)}{2\varepsilon p_{T^\omega_{z+\omega'(u)}+u}(-z)}.$$

The third integral may be rewritten as

$$\frac{P(|T^\omega_{z+\omega'(u)}+u-1|<\varepsilon)}{2\varepsilon}E^{(-z-\omega'(u),\infty)}\left[\frac{H(\omega)}{p_{T^\omega_{z+\omega'(u)}+u}(-z)}\middle| |T^\omega_{z+\omega'(u)}+u-1|<\varepsilon\right],$$

with a slightly improper writing (the ω's should not appear in the expectation, but we keep them to keep the distinction with the expectation with respect to ω'). Similar arguments as above imply that the limit we are looking for is

$$P^1_{0\to -z}(H(\underleftarrow{X})H'(\underrightarrow{X})) = p_1(-z)^{-1}\int_0^1 \mathrm{d}u N^{>u}\left[H'(\omega')q_{z+\omega'(u)}(1-u)E^{1-u}_{0\downarrow -(z+\omega'(u))}[H(\omega)]\right].$$

This in turn completely determines the law of the bridge by a monotone class argument. A careful application of the above identity thus gives

$$E^1_{0\to -z}[f(1-(\tau_3-\tau_1))] = p_1(-z)^{-1}\int_0^1 \mathrm{d}u N^{>u}\left[q_{z+\omega'(u)}(1-u)E^{1-u}_{0\downarrow -(z+\omega'(u))}[f(u+T^\omega_{\omega'(u)})]\right].$$

Applying Lemma 15 to the rightmost expectation term, this is equal to

$$\begin{aligned}
&p_1(-z)^{-1}\int_0^1 \mathrm{d}u N^{>u}\left[q_{z+\omega'(u)}(1-u)\int_0^{1-u}\mathrm{d}v\frac{q_{\omega'(u)}(v)q_z(1-u-v)}{q_{\omega'(u)+z}(1-u)}f(u+v)\right]\\
&= p_1(-z)^{-1}\int_0^1 \mathrm{d}u\int_u^1 \mathrm{d}s f(s)q_z(1-s)N^{>u}\left[q_{\omega'(u)}(s-u)\right]\\
&= zq_z(1)^{-1}\int_0^1 \mathrm{d}s f(s)q_z(1-s)\int_0^s \mathrm{d}u N^{>u}\left[q_{\omega'(u)}(s-u)\right]
\end{aligned}$$

It remains to compute the second integral. Using scaling identities for $N^{>u}$ and $q_x(s)$ we have

$$\begin{aligned}
\int_0^s \mathrm{d}u N^{>u}\left[q_{\omega'(u)}(s-u)\right] &= \int_0^1 \mathrm{d}r N^{>sr}\left[q_{\omega'(sr)}(s(1-r))\right]\\
&= s^{-1/\alpha}\int_0^1 \mathrm{d}r s^{1/\alpha}N^{>sr}\left[q_{s^{-1/\alpha}\omega'(sr)}(1-r)\right]\\
&= s^{-1/\alpha}\int_0^1 \mathrm{d}r N^{>r}\left[q_{\omega'(r)}(1-r)\right].
\end{aligned}$$

Finally, the integral in the right hand side does not depend on s, we call it c and obtain

$$E^1_{0\to -z}[f(1-(\tau_3-\tau_1))] = \int_0^1 \mathrm{d}s f(s)\frac{czq_z(1-s)}{s^{1/\alpha}q_z(1)}.$$

So we necessarily have $c = c_\alpha$, and the claim follows. $\square$

Proof of Proposition 4. The proof is now easily obtained by combining the last lemmas. Under $\mathbf{P}^1_{0\to 0}$, conditionally on $Z^{(t)}_1 = z$, the law of the lengths of constancy intervals of $\underline{V^{\tau_2}X^{(t)}}$ is obtained by adjoining the term $1-(\tau_3-\tau_1)$ to a sequence which, conditionally on $1-(\tau_3-\tau_1)=t$, has same law as $\Delta T_{[0,z]}$ given $T_z = 1-t$ under P (Lemma 17). By Lemma 18, $1-(\tau_3-\tau_1)$ has itself the law of a size-biased pick from $\Delta T_{[0,z]}$ given $T_z = 1$ under P, so Lemma 1 shows the whole sequence has the law of $\Delta T_{[0,z]}$ given $T_z = 1$. Last, by Lemma 3, $Z^{(t)}_1$ has density $p^{(t)}_1(-z)\rho^{(t)}_1(z)p_1(0)^{-1}\mathrm{d}z$, entailing the claim. $\square$

5.3 Proof of Theorem 2

To recover the dislocation measure of $F^\natural$, we use the following variation of Lemma 10 and [22, Corollary 1]. For details on size-biased versions of measures on S, see e.g. [13], which deals with probability measures, but the results we mention are easily extended to σ-finite measures.

Proposition 5 *Let $(F(t), t\geq 0)$ be a ranked self-similar fragmentation with characteristics $(\beta, 0, \nu)$, $\beta \geq 0$. For every t, let $F_*(t)$ be a random size-biased permutation of the sequence $F(t)$ (defined on a possibly enlarged probability space). Let G be a continuous bounded function on the set of non-negative sequences with sum ≤ 1, depending only on the first I terms of the sequence, with support included in a set of the form $\{s_i \in [\eta, 1-\eta], 1\leq i\leq I\}$. Then*

$$\frac{1}{t}E[G(F_*(t)] \underset{t\downarrow 0}{\rightarrow} \nu_*(G),$$

where ν_* is the size-biased version of ν characterized by

$$\nu_*(G) = \int_S \nu(\mathrm{d}\mathbf{s}) \sum_{j_1,\ldots,j_I} G(s_{j_1},\ldots,s_{j_I}) s_{j_1}\frac{s_{j_2}}{1-s_{j_1}}\cdots\frac{s_{j_I}}{1-s_{j_1}-\ldots-s_{j_I}},$$

where the sum is on all possible distinct $j_1,\ldots,j_I$. Moreover, ν can be recovered from ν_*.

Proof of Theorem 2. Let G be a function of the form $G(x) = f_1(x_1)\ldots f_I(x_k)$ for $x=(x_1,x_2,\ldots)$ and $\sum_i x_i \leq 1$, with $f_1,\ldots,f_I$ continuous bounded functions on $[0,1]$ that are null on a set of the form $[0,1]\backslash]\eta,1-\eta[$. Let $\Delta^* T_{[0,z]}$ be the sequence of the jumps of T on the interval $[0,z]$, listed in size-biased order (which involves some enlargement of the probability space). Using Lemma 1, it is easy that $z\mapsto E[G(\Delta^* T_{[0,z]})|T_z = 1]$ is a continuously differentiable function with derivative bounded by some $M>0$. Let also $F^\natural_*(t)$ be the sequence $F^\natural(t)$ listed in size-biased order. Now by Proposition 4,

$$\mathbf{N}^{(1)}\left[\frac{G(F^\natural_*(t))}{t}\right] = \frac{1}{t}\mathbf{E}\left[e^{-t^\alpha + tZ^{(t)}_1} p_1(-Z^{(t)}_1)p_1(0)^{-1}\widetilde{E}\left[G(\Delta^*\widetilde{T}_{[0,Z^{(t)}_1]})\,\Big|\widetilde{T}_{Z^{(t)}_1} = 1\right]\right],$$

where $\widetilde{T}$ is a copy of T with law $\widetilde{E}$, independent of the marked process X. Consider a function $f(t,z)$ that is continuous in t and x and null at $(t,0)$ for every $t\geq 0$. Then the compensation formula applied the subordinator $Z^{(t)}$ between times 0 and 1 gives

$$\begin{aligned}\frac{1}{t}\mathbf{E}[f(t,Z^{(t)}_1)] &= \frac{1}{t}\int_0^1 \mathrm{d}x \int C_\alpha(1-e^{-ts})s^{-\alpha-1}\mathrm{d}s\mathbf{E}[f(t,Z^{(t)}_x+s)-f(t,Z^{(t)}_x)] \\ &\underset{t\to 0}{\rightarrow} C_\alpha\int_0^1 \mathrm{d}x\int s^{-\alpha}\,\mathrm{d}s\, f(0,s) = C_\alpha\int s^{-\alpha}\mathrm{d}s f(0,s),\end{aligned}$$

as soon as we may justify the convergence above. Take

$$f(t,z) = \exp(-t^\alpha + tz)p_1(-z)p_1(0)^{-1}E[G(\Delta T^*_{[0,z]})|T_z = 1],$$

then we have to check that $s^{-\alpha}\mathbf{E}[|f(t, Z_x^{(t)} + s) - f(t, Z_x^{(t)})|]$ is bounded independently on $x \in [0,1]$. By the hypotheses on G, it is again true that $z \mapsto f(t,z)$ is a continuously differentiable function with uniformly bounded derivative, when t stays in a neighborhood of 0. Hence the expectation above is bounded by $(M's \wedge M'')s^{-\alpha}$ for some $M', M'' > 0$, which allows to apply the dominated convergence theorem. By Proposition 5, we obtain, denoting by $\nu_\natural$ the dislocation measure of $F^\natural$,

$$\begin{aligned} t^{-1}\mathbf{N}^{(1)}[G(F^\natural_*(t))] \quad &\underset{t\to 0}{\to} \quad \int_S \nu_\natural(\mathrm{d}\mathbf{s})G(s_{j_1},\ldots,s_{j_I}) \sum_{j_1,\ldots,j_I} s_{j_1}\frac{s_{j_2}}{1-s_{i_1}}\cdots\frac{s_{j_I}}{1-s_{j_1}-\ldots-s_{j_{I-1}}} \\ &= \quad C_\alpha \int_0^\infty \mathrm{d}s \frac{s^{-\alpha}p_1(-s)}{p_1(0)} E[G(\Delta T^*_{[0,s]})|T_s = 1], \end{aligned}$$

allowing to conclude that $\nu_\natural = \nu_+$ with the same computations as in the proof of Lemma 12. □

6 Asymptotics

In this section we discuss asymptotic results for F^+.

6.1 Small-time asymptotics

Proposition 6 *Let Z be a non-negative stable $(\alpha-1)$ random variable with Laplace transform $E[\exp(-\lambda Z)] = \exp(-\alpha\lambda^{\alpha-1})$. Denote by $\Delta_1, \Delta_2, \ldots$ the ranked jumps of $(T_x, 0 \leq x \leq Z)$, where T is as before the stable $1/\alpha$ subordinator, which is taken independent of Z. Then*

$$t^{\alpha/(1-\alpha)}(F_2^+(t), F_3^+(t), \ldots) \underset{t\to 0+}{\overset{d}{\to}} (\Delta_1, \Delta_2, \ldots).$$

We first need the

Lemma 19 *Let $Z_1^{(t)}$ have the law $\rho_1^{(t)}(s)\mathrm{d}s$ above, then*

$$t^{1/(1-\alpha)} Z_1^{(t)} \underset{t\to 0+}{\overset{d}{\to}} Z,$$

where Z is as above a stable variable with Laplace exponent $\alpha\lambda^{\alpha-1}$.

Proof. Recall that $Z^{(t)}$ is a subordinator with characteristic exponent given by

$$\mathbf{E}[e^{-\lambda Z_1^{(t)}}] = \exp\left(-\int_0^\infty \frac{C_\alpha(1-e^{-tx})\mathrm{d}x}{x^{\alpha+1}}(1-e^{-\lambda x})\right).$$

Therefore, evaluating the Laplace exponent at the point $t^{1/(1-\alpha)}\lambda$, changing variables and using dominated convergence entails

$$\mathbf{E}[\exp(-\lambda t^{1/(1-\alpha)}Z_1^{(t)})] \underset{t\to 0+}{\to} \exp\left(-\int_0^\infty \frac{C_\alpha \mathrm{d}y}{y^\alpha}(1-e^{-\lambda y})\right).$$

Thus the convergence to some limiting Z. Using now the explicit value for C_α, we see that the Laplace exponent of Z has to be $\alpha\lambda^{\alpha-1}$, as claimed. $\square$

The proof of Proposition 6 follows the same lines as for Proposition 6 in [21], so we will only sketch it. One first begins with proving that if $\mathcal{Z}$ is as in Lemma 3 a random variable distributed according to the law that has density $\rho_1^{(t)}(z)p_1^{(t)}(-z)\mathrm{d}z/p_1(0)$, then $t^{1/(1-\alpha)}\mathcal{Z}$ converges in law to Z. This is a consequence of the preceding lemma, since as $t \to 0$, $X^{(t)}$ converges to X, so one can write

$$\mathbf{E}[g(t^{1/(1-\alpha)}\mathcal{Z})] = \mathbf{E}[g(t^{1/(1-\alpha)}Z_1^{(t)})p_1^{(t)}(-Z_1^{(t)})/p_1(0)],$$

where $Z_1^{(t)}$ is distributed as above. By Skorokhod's representation theorem, we may suppose that $t^{1/(1-\alpha)}Z_1^{(t)}$ converges a.s. to its limit in law Z, So it remains to show that a.s. $p_1^{(t)}(-Z_1^{(t)}) \to p_1(0)$ as $t \to 0$ to apply dominated convergence, and this is done by recalling that $p_1^{(t)}(z) = e^{-t^\alpha - tz}p_1(z)$. Then one reasons by induction just as in [21, Proposition 6], using the explicit form of the semigroup of F^+.

6.2 Large-time asymptotics

By a direct application of Theorem 3 in [10], one gets the large t asymptotic behavior for F^+. Recall that the Gamma law with parameter a is the law with density proportional to $x^{a-1}e^{-x}$ on $\mathbb{R}_+$. The moments of this law are given, for $r > -a$, by

$$\frac{1}{\Gamma(a)}\int_0^\infty x^{r+a-1}e^{-x}\mathrm{d}x = \frac{\Gamma(a+r)}{\Gamma(a)}.$$

Proposition 7 *Define*

$$\rho_t(\mathrm{d}y) = \sum_{i=1}^\infty F_i(t)\delta_{t^\alpha F_i(t)}(\mathrm{d}y),$$

then ρ_t is a probability measure that converges in law as $t \to \infty$ to the deterministic Gamma law with parameter $1 - 1/\alpha$.

Proof. We know by [10, Theorem 3] that ρ_t converges to some probability ρ_∞ that is characterized by its moments,

$$\int_0^\infty y^{k/\alpha}\rho_\infty(\mathrm{d}y) = \frac{\alpha(k-1)!}{\Phi'(0+)\Phi\left(\frac{1}{\alpha}\right)\ldots\Phi\left(\frac{k-1}{\alpha}\right)}$$

for every $k \geq 1$, where Φ is the Laplace exponent of a subordinator related to a tagged fragment of the process F^+. This exponent depends only on the dislocation measure (and not the index), so it is the same as for F_- in [21]. By taking the explicit value of Φ (Section 3.2 therein), we easily get

$$\int_0^\infty y^{k/\alpha}\rho_\infty(\mathrm{d}y) = \left(\frac{\alpha\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}\right)^k \frac{\Gamma\left(1+\frac{k-1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}\right)} = \frac{\Gamma\left(1+\frac{k-1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}\right)}.$$

Replacing k by αk, one can recognize the moments of the Gamma law with the claimed parameter. $\square$

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